Clique-independent sets of Helly circular-arc graphs

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Abstract

A circular-arc graph is the intersection graph of arcs of a circle. A Helly circular-arc graph is a circular-arc graph admitting a model whose arcs satisfy the Helly property. A clique-independent set of a graph is a set of pairwise disjoint cliques of the graph. It is NP-hard to compute the maximum cardinality of a clique-independent set for a general graph. In the present paper, we propose algorithms for finding the maximum cardinality and weight of a clique-independent set of a $\overline{3K_2}$-free CA graph. Also, we apply the algorithms to the special case of an HCA graph. The complexity of the proposed algorithm for the cardinality problem in HCA graphs is $O(n)$. This represents an improvement over the existing algorithm, whose complexity is $O(n^2)$.

Key words: algorithms, circular-arc graphs, clique-independent sets, Helly circular-arc graphs

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1 Introduction

Let $G = (V(G), E(G))$ be an undirected connected graph, $|V(G)| = n$ and $|E(G)| = m$. A *clique* is a complete subgraph maximal under inclusion. A *clique-independent set* is a set of pairwise disjoint cliques. Write $\alpha(G)$ and $\alpha_c(G)$ to represent the maximum cardinality of an independent set and a clique-independent set of G, respectively. When weights are considered, denote by $\tilde{\alpha}(G)$ and $\tilde{\alpha}_c(G)$ these corresponding parameters, respectively. The first NP-hardness result for clique-independent sets appears in [1].

A *circular-arc* (CA) *model* for G is a pair $(C, \mathcal{A})$, where C is a circle and $\mathcal{A}$ is a collection of arcs of C, such that each arc A_i of $\mathcal{A}$ corresponds to a vertex v_i of $V(G)$, and A_i, A_j intersect precisely when v_i, v_j are adjacent, for distinct i, j. A *circular-arc* (CA) *graph* is one admitting a CA model. When traversing the circle C, we will always choose the clockwise direction. If s, t are points of C, write (s, t) to mean the arc of C defined by traversing the circle from s to t. Call s, t the extremes of (s, t), while s is the start and t the end of the arc. For arc A_i of $\mathcal{A}$, write $A_i = (s_i, t_i)$.

A *Helly circular-arc* (HCA) *graph* G is a CA graph admitting a CA model whose arcs satisfy the Helly property. That is, every pairwise intersecting subfamily of arcs of $\mathcal{A}$ contains a common point. On the other hand, $\overline{3K_2}$-free circular-arc graphs contain Helly circular-arc graphs.

In the present paper, we propose algorithms for finding the maximum cardinality and weight of a clique-independent set of a $\overline{3K_2}$-free CA graph. Also, we apply the algorithms to the special case of an HCA graph. The complexity of the proposed algorithm for the cardinality problem in HCA graphs is $O(n)$, an improvement over the algorithm by [3], which has complexity $O(n^2)$.

Let G be a graph admitting a CA model $(C, \mathcal{A})$. For an arc A_i of $\mathcal{A}$, $V(A_i)$ is the vertex of G corresponding to A_i. Similarly, for a subset $\mathcal{A}'$ of arcs of A, $V(\mathcal{A}')$ denotes the collection of vertices of G corresponding to the arcs of $\mathcal{A}'$. If an arc A_i of $\mathcal{A}$ contains the point p of C then say that A_i is an arc of p. Denote by $\mathcal{A}(p)$ the collection of arcs of p. Clearly, $V(\mathcal{A}(p))$ is a complete set of G. For points p, p' of C, say that p (properly) dominates p' when $\mathcal{A}(p)$ (properly) contains $\mathcal{A}(p')$. When $\mathcal{A}(p) = \mathcal{A}(p')$ then p, p' are equivalent. Say that p is a complete point when no point of C properly dominates p. In addition when $V(\mathcal{A}(p))$ is a clique of G then p is a clique point of C. Such a clique is called a Helly clique. If $(C, \mathcal{A})$ is a Helly model then all its cliques are Helly and there is a 1-1 correspondence between cliques of G and non equivalent clique points of C. On the other hand, any non Helly clique contains at least three vertices. Also, among the arcs of A corresponding to the vertices of a non Helly clique there exist always three of them which together cover the entire circle.

2 Intersection Segments

A method for finding sets of complete points of a CA graph is discussed. These sets will be employed in the proposed algorithms. Let G be a graph admitting a CA model $(C, \mathcal{A})$. Let $A = A_1, \ldots, A_n$ and $A_i = (s_i, t_i)$. A *segment* is an arc of C formed by two consecutive extremes of the arcs of $\mathcal{A}$, when traversing C. There are $2n$ segments, which exactly cover C, except for their extreme points. An *intersection segment* is a segment of the type (s_i, t_j), that is, its start point is the start point of some arc arc A_i of $\mathcal{A}$, while its end point is the end of an arc A_j of A. Denote $I_i = (si, tj)$. A *point* p_i of I_i is called an intersection point. The following theorem relates complete points to intersection points.

Theorem 1 *Every complete point is an intersection point.*

An intersection segment $I_i = (s_i, t_j)$ is *simple* when A_i and A_j together do not cover the circle C and *universal* otherwise. A point belonging to a simple segment is a *simple point*, whereas one inside a universal segment is a *universal point*. The next theorem relates intersection points to complete points.

Theorem 2 *Let $I_i = (s_i, t_j)$ be an intersection segment and p a point of I_i. If p is simple then p is complete. If p is universal then p is either complete or it is properly dominated by some intersection point p' belonging to (s_j, t_i).*

Next, we consider some special subsets of points of C. A *complete (intersection) (simple) (universal) (clique) point representation* of C is a maximal set of complete (intersection) (simple) (universal) (clique) non equivalent points of C. Represent these sets by P, T, S, U, Q, respectively. Let P', P'' be two subsets of points of C. Then P', P'' are isomorphic when there exists a bijection f between these sets such that p' and $f(p')$ are equivalent, for all points p' belonging to P'. Sets P, T, S, U, Q are all unique, up to isomorphism. Consequently, we can write T as the union of S and U, and P as the union of S and U', where U' is a convenient subset of U. Also, Q is a subset of P, with $Q = P$ precisely when (C, A) is a Helly model. Clearly, Q corresponds to the set of Helly cliques of G. Moreover, the Helly cliques can be further bipartitioned, as follows. Let M_i be a Helly clique of G and p_i the clique point of Q corresponding to M_i. Then M_i is a simple clique or universal clique, according whether p_i is a simple or universal point, respectively.

Theorems 1 and 2 lead to an algorithm for constructing a complete point representation P of C, given a CA model $(C, \mathcal{A})$ for a graph G. In fact, the algorithm constructs explicitly the simple and universal point representations S and U, respectively, and then finds the subset U' of U, such that P is the union of S and U'. The algorithm is divided into two steps. Step 1 constructs S and U, while Step 2 determines U' by including in it one universal point (the one with lowest index), for each collection of equivalent complete points.

Algorithm 1: CONSTRUCTING A COMPLETE POINT REPRESENTATION OF A CA GRAPH

STEP 1: Identify the segments of C. Define S and U as equal the empty set. For each segment (p_1, p_2) of C, if p_1 is the start of some arc A_i of $\mathcal{A}$ and p_2 is the end of the arc A_j contained in A, then let p_i be a point of (p_1, p_2) and perform the the additional test: if A_i and A_j cover C then include p_i in U, otherwise include p_i in S. At the end, T is the union of S and U.

STEP 2: Define U' as the empty set. For each universal point p_i of U, let $I_i = (s_i, t_j)$ be its corresponding universal segment. For each point p_i of U, apply the following procedure. Compute $\mathcal{A}(p_i)$. Define $L = i$. Traverse the arc (s_j, t_i) of C, segment by segment, in the order as they appear. In case an intersection segment (s_k, t_l) is found choose a point p_k of (s_k, t_l), compute $\mathcal{A}(p_k)$, and if $\mathcal{A}(p_i) = \mathcal{A}(p_k)$ then include k in L. After all segments contained in (s_j, t_i) have been traversed then include p_i in U' precisely in the case where p_i is not properly dominated by any such p_k and i is the minimum k such that p_k belongs to L. At the end, P is the union of S and U'. $\triangle$

It is not difficult to verify that the algorithm is correct. We analyse its complexity. Algorithm 1 constructs S and U in $O(n)$ time and the subset U' of U in $O(n^2)$ time. Consequently, we require $O(n^2)$ time for constructing P. Furthermore Q can be constructed in $O(n^2)$ time, which implies that we can generate all cliques of an HCA graph in $O(n^2)$ time.

3 Clique-Independent Sets

In this section, we describe algorithms for finding the maximum cardinality and weight of a clique-independent set of a $\overline{3K_2}$-free CA graph, and apply the algorithms to the special case of an HCA graph. Let G be a graph admitting a CA model $(C, \mathcal{A})$. Let $I_i = (s_i, t_j)$ be an intersection segment of I_i and $p_i \in I_i$. The *arc reduction of* p_i is the arc $R_i = (s_i, t_a) \subseteq C$, where $(p_i, t_a) \supseteq (p_i, t_l)$, for all $A_l \in \mathcal{A}(p_i)$. That is, for $I_i = (s_i, t_j)$, select the arc $A_a = (s_a, t_a) \in \mathcal{A}(p_i)$, such that t_a is as far as possible from t_j, when traversing C. Then $R_i = (s_i, t_a) \subseteq A_a$. Let Q be the clique point representation of C, and $Q' \subseteq Q$. Denote by $G_R(Q')$ the intersection graph of the arc reductions of the clique points of Q'. On the other hand, denote by $\hat{K}(Q')$ the intersection graph of the family of subsets of arcs $\{\mathcal{A}(p_i) | p_i \in Q'\}$. The following theorem relates these two graphs.

Theorem 3 *Let G be a graph admitting a CA model $(C, \mathcal{A})$, Q the clique point representation of C and $Q' \subseteq Q$. Then $G_R(Q') \cong \hat{K}(Q')$.*

We need $O(n^2)$ time for the construction of Q, while the all arc reductions requires in $O(n)$ time. Also, it can be seen that the construction of the clique graph of G (the intersection graph of the cliques of G) takes $O(n^2)$ time.

There is a straightforward relation between clique-independent sets of a graph G and independent sets of its clique graph.

Theorem 4 *Let G be a CA graph, Q its clique point representation, $Q' \subseteq Q$, $V' \subseteq V(\hat{K}(Q'))$ and $\mathcal{M}'$ the set of cliques of G corresponding to those clique points of Q' associated to the vertices of V' in $\hat{K}(Q')$. Then $\mathcal{M}'$ is a clique-independent set of G if and only if V' is an independent set of $\hat{K}(Q')$.*

Next theorem describes conditions which would force two cliques to intersect.

Theorem 5 *Let G be a graph with a CA model $(C, \mathcal{A})$, and M, M' cliques of it. If M, M' satisfy any of the conditions below then $M \cap M' \neq \emptyset$.*

(1) M is Helly and M' is universal
(2) M is non Helly and M' is Helly
(3) M and M' are both non Helly and G is $\overline{3K_2}$-free

The above proposition lead to the following equation for computing the clique-independence number of a $\overline{3K_2}$-free CA graph.

Theorem 6 *Let G be a $\overline{3K_2}$-free CA graph, S and Q the simple and clique point representations, relative to some CA model of G, respectively. Then $\alpha_c(G)$ is equal to 1, when $S \cap Q = \emptyset$, and equal to $\alpha(G_R(S \cap Q))$, otherwise.*

The proposed algorithm for determining $\alpha_c(G)$ corresponds to the computation of the equation given by Theorem 6. Let G be a $\overline{3K_2}$-free graph with a given CA model.

Algorithm 2: CLIQUE-INDEPENDENCE NUMBER OF A $\overline{3K_2}$-FREE CA GRAPH

First, construct the simple and clique point representations S and Q, respectively. If $S \cap Q = \emptyset$ then $\alpha_c(G) = 1$. Otherwise, find all arc reductions and construct a CA model for the graph $G_R(S \cap Q)$. Finally, find the maximum independent set of $G_R(S \cap Q)$, as $\alpha_c(G) = \alpha(G_R(S \cap Q))$. $\triangle$

Finding S requires $O(n)$ time by Algorithm 1, but the determination of Q takes $O(n^2)$ time. The construction of all arc reductions and finding the CA model of $G_R(S \cap Q)$ can be done in $O(n)$ time. Finally, the maximum independent set of $G_R(S \cap Q)$ can be computed in $O(n)$ time [2]. Therefore the overall complexity of Algorithm 2 is $O(n^2)$. In particular, if the given model is HCA then all simple points are clique points. So, the complexity reduces to $O(n)$.

The solution of the weighted clique-independence problem is similar. Let G be a $\overline{3K_2}$-free CA graph, where there is a weight assigned to each vertex. For a clique M of G, define its weight $\tilde{M}$ as the sum of the weights of the vertices which form M. The equation below determines the maximum weight of a clique-independent set of G.

Theorem 7 *Let G be a $\overline{3K_2}$-free CA graph, S and Q the simple and clique point representations, relative to some CA model of G, respectively. Denote by $\tilde{M}_1$ the maximum weight of a single clique of G. Then $\tilde{\alpha}_c(G)$ is equal to $\tilde{M}_1$, and equal to $max\{\tilde{M}_1, \tilde{\alpha}(G_R(S \cap Q))\}$, otherwise.*

The algorithm for computing the above formula is as follows. Let G be a $\overline{3K_2}$-free graph with a given CA model.

Algorithm 3: MAXIMUM WEIGHT OF A CLIQUE-INDEPENDENT SET OF A $\overline{3K_2}$-FREE CA GRAPH

Generate all the cliques of G and select the clique M_1 having maximum weight $\tilde{M}_1$. Construct S and Q. If $S \cap Q = \emptyset$ then $\tilde{\alpha}_c(G) = \tilde{M}_1$. Otherwise, construct the CA model of $G_R(S \cap Q)$ and $\tilde{\alpha}_c(G) = max\{\tilde{M}_1, \tilde{\alpha}(G_R(S \cap Q))\}$. $\triangle$

When G is $\overline{3K_2}$-free, G has at most $O(n^4)$ cliques [4]. Consequently, the cliques of G can be generated in $O(n^5 m)$ time, using algorithm [6]. The weighted independent set of a CA graph can be determined in $O(n^2)$ time [5]. The remaining operations require $O(n)$ time. The overall complexity of Algorithm 3 is therefore $O(n^5 m)$. When G admits an HCA model, G has at most n cliques. Consequently, the complexity of the algorithm reduces to $O(n^2)$.

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