# WARDROP EQUILIBRIA

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A common behavioral assumption in the study of transportation and telecommunication networks is that travelers or packets, respectively, choose routes that they perceive as being the shortest under the prevailing traffic conditions [1]. The situation resulting from these individual decisions is one in which drivers cannot reduce their journey times by unilaterally choosing another route, which prompted Knight [2] to call the resulting traffic pattern an equilibrium. Nowadays, it is indeed known as the *Wardrop (or user) equilibrium* [3], and it is effectively thought of as a steady state evolving after a transient phase in which travelers successively adjust their route choices until a situation with stable route travel costs and route flows has been reached [4]. In a seminal contribution, Wardrop [5, p. 345] stated two principles that formalize this notion of equilibrium and the alternative postulate of the minimization of the total travel costs. His first principle reads:

> The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Wardrop's first principle of route choice, which is identical to the notion postulated by Kohl [1] and Knight [2], became accepted as a sound and simple behavioral principle to describe the spreading of trips over alternate routes due to congested conditions [6].

Since its introduction in the context of transportation networks in 1952 and its mathematical formalization by Beckmann *et al*. [7], transportation planners have been using Wardrop equilibrium models to predict commuters decisions in real-life networks [6,8]. These models have been and are still used today to evaluate alternative future scenarios and decide a route of actions. Typical examples include allocation of investment for capacity expansion when building roads and bridges, optimizing the value of tolls, and making policy decisions. The work of Korolis *et al*. [9] introduced these concepts in telecommunication research, which have become popular ever since [10].

## THE BASIC MODEL

An instance of the *traffic assignment problem* is given by the transportation offer—represented by the network topology, road geometry, road capacity, and arc link travel cost functions—and the transportation demand—represented by the list of origin–destination (OD) pairs and their demand rates.

We consider a directed network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, and a set $\mathcal{C} \subseteq \mathcal{N} \times \mathcal{N}$ of commodities represented by OD pairs. For each $k \in \mathcal{C}$, a flow of demand rate equal to $d_k$ must be routed from the corresponding origin to its destination. The basic model assumes that demands are arbitrarily divisible; in fact, the routing decision of a single individual has only an infinitesimal impact on other users. For $k \in \mathcal{C}$, let $\mathcal{R}_k$ be the set of routes in $\mathcal{G}$ connecting the corresponding origin and destination, and let $\mathcal{R} := \bigcup_{k\in\mathcal{C}} \mathcal{R}_k$. A link flow is a nonnegative vector $f = (f_a)_{a\in\mathcal{A}}$ describing the traffic rate in each link. Furthermore, a nonnegative, nondecreasing, and continuous link travel cost function $t_a(\cdot)$, with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$, maps the flow $f_a$ on arc $a$ to the time needed to traverse $a$. A route flow is a nonnegative vector $h = (h_r)_{r\in\mathcal{R}}$ that meets the demand, that is, $\sum_{r\in\mathcal{R}_k} h_r = d_k$ for $k \in \mathcal{C}$. Given a route flow, the corresponding link flow is easily computed as $f_a = \sum_{r\ni a} h_r$, for each $a \in \mathcal{A}$. For a flow $f$, the travel cost along

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a route $r$ is $c_r(f) := \sum_{a \in r} t_a(f_a)$. Let $X$ be the set of feasible flows $(f, h)$ and $X_f$ its projection into the space of arc flows. Note that this set is a polytope given simply by the flow-conservation constraints for each commodity on every node (see ***Multicommodity Flows*** and Ahuja *et al*. [11]).

Interpreting Wardrop's first principle as requiring that all flow travels along shortest paths, a flow $h$ is called a *Wardrop equilibrium* if and only if for all $k \in \mathcal{C}$, we have that

$$c_r(h) = \min_{q \in \mathcal{R}_k} c_q(h), \tag{1}$$

for all $r \in \mathcal{R}_k$ such that $h_r > 0$. Beckmann *et al*. [7] proved that such a flow always exists by considering the following min-cost multicommodity flow problem with separable objective function:

$$\min \left\{ \sum_{a \in \mathcal{A}} \int_0^{f_a} t_a(z)\, \mathrm{d}z : f \in X_f \right\}. \tag{2}$$

The previous problem is convex because the objective is the integral of a nondecreasing function and, since its domain is a compact set, attains its optimum (for some background on convex optimization. Actually, it can be proved that its first-order optimality conditions are

$$c_r(h) \leq c_q(h), \quad \text{for all } k \in \mathcal{C} \text{ and all routes } r, q \in \mathcal{R}_k \text{ such that } h_r > 0, \tag{3}$$

which is equivalent to Equation (1).

If cost functions $t_a$ are strictly increasing, $f$ is unique (but there can be different flow decompositions $h$). For the case of nondecreasing costs, the vector of costs $(t_a(f_a))_{a \in \mathcal{A}}$ is unique under the possibly nonunique equilibria. Computationally, Equation (2) implies that an equilibrium can be computed efficiently using general convex optimization techniques (see the section titled “Computation of Wardrop Equilibria”)[1]. A formulation similar to the one presented here can be used to perform a sensitivity analysis of Wardrop equilibria, which can be useful when testing the robustness of the model [12].

Yet another important characterization of traffic equilibrium problems, due to Smith [13], consists of reformulating Equation (2) as a variational inequality problem, see also Dafermos [14] and ***Variational Inequalities***. Accordingly, a flow $f$ is a user equilibrium if and only if

$$\sum_{a \in \mathcal{A}} t_a(f_a) f_a \leq \sum_{a \in \mathcal{A}} t_a(f_a) x_a, \quad \text{for all flows } x \in X_f. \tag{4}$$

Note that this inequality is a direct consequence of the fact that in equilibrium, users travel on shortest paths with respect to arc costs $t_a(f_a)$. Another reformulation that is useful to characterize equilibria in very general settings is given by nonlinear complementary problems [15] (see ***Complementarity Problems***).

Charnes and Cooper [16] were the first to notice that the concepts of Nash and Wardrop equilibria are related. Haurie and Marcotte [17] prove that a Nash equilibrium in a network game with a finite number of players converges to a Wardrop equilibrium when the number of players increases (for some background on game theory, see the section titled “Noncooperative Strategic-Form Games” in this encyclopedia). For this reason, although the solution concepts are different, a Wardrop equilibrium can be viewed as an instance of a Nash equilibrium in a game with a large number of players. De Palma and Nesterov [18] look at generalizations and alternative definitions of the basic model and established conditions that guarantee the existence of equilibria. For example, Wardrop equilibria still exist if cost functions are only required to be lower semicontinuous. Marcotte and Patriksson [19] also discuss alternative definitions of equilibria in network games and the relationships between them.

[1]In the case of nonmonotone costs, the solutions to Equation (2) are also equilibria. In this case, though, uniqueness is not guaranteed even as flow on arcs because Equation (2) is not necessarily a convex problem and hence, may admit multiple local optima.

To conclude, let us note that in practice a transportation planner needs to find or estimate all the elements that comprise the model. The topology of the network is usually digitized from maps, if it is not already available. Link travel cost functions are calibrated from historical information using tabulated functions that relate geometry of the road to capacity [20,21]. One may need to also add tolls or other generalized costs to the arcs, which can be converted to the same units by using the average value of time for the population. The latter can usually be estimated from socioeconomic information coming from census data. Demand can be measured directly or may come from historical OD matrices that can be calibrated using up-to-date traffic counts [8].

## MORE GENERAL MODELS

Many extensions of the model presented in the previous section have been analyzed and studied since the introduction of the Wardrop equilibrium model in 1952. We now present a selection of extensions we believe are particularly interesting.

An important generalization consists in allowing link travel cost functions to depend on the full vector of flows. In that case, the function that represents congestion can be encoded by the operator $t(f) : \mathbb{R}^{\mathcal{A}}_{\geq 0} \to \mathbb{R}^{\mathcal{A}}_{\geq 0}$. This is of practical relevance because the cost in one arc usually depends on the load in other arcs. Typical examples in the area of transportation modeling include representing intersections more accurately when cross streets influence each other, and two-way streets where traffic going one way can impact the reverse lanes. Furthermore, to consider different vehicle types interacting in the same network, one can create one copy of the network per vehicle type and have congestion depend on the full load, defined as the weighted sum across all copies. An example in the area of wireless telecommunications is interference, which can make delays of nearby cells grow. In the nonseparable case, it is convenient to require that the operator $t$ is monotone, meaning that $(t(f) - t(f')) \cdot (f - f') \geq 0$, for all $f, f' \in \mathbb{R}^{\mathcal{A}}_{\geq 0}$, a generalization of monotonicity in the separable case. We say that costs are nonseparable, symmetric when for any two arcs, the influence of traffic in one arc to the congestion on the other is equal to the reverse influence (with symbols, $\partial t_a(f)/\partial f_b = \partial t_b(f)/\partial f_a$, for all $a, b \in \mathcal{A}$ and all $f \in \mathbb{R}^{\mathcal{A}}_{\geq 0}$). In that case, the integral in Equation (2) can be replaced by $\int_0^f t(z)\,\mathrm{d}z$, which is well defined, and an equilibrium can still be computed using convex optimization techniques. An easy example of this case is when $t(f) = \Theta f + \theta$, with $\Theta$ a symmetric matrix of dimension $|\mathcal{A}|$. It is worth noting that this symmetry condition is equivalent to the requirement that the game is *potential* with a continuum of agents [22] (for an introduction to potential games, the reader is referred to Monderer and Shapley [23]). In the general asymmetric case, the equilibrium cannot be formulated as a convex optimization problem (but note that there are exceptions and in some cases they can be formulated as nondifferentiable optimization problems [24]) and one has to resort to abstractions such as variational inequalities and nonlinear complementarity problems. Indeed, Equation (4) still holds and can be used to prove existence and uniqueness results, as well as a way to compute equilibria.

Another important extension to the basic model that goes in a different direction is referred to by elastic demands. A network model normally represents a spectrum of options that a user of the system has. In reality, a user will normally have more alternatives than those present in a model. For example, in a car transportation model, some users may decide to take a subway when roads are too congested or cancel the trip when the combination of travel time, tolls, and gas exceeds the utility of the trip. This can be captured by a model with a demand of users that will participate only when their willingness to pay is higher than the cost of trips. In this case, the link travel cost functions in the equilibrium model are complemented by a demand function $\delta_k(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ per OD pair $k \in \mathcal{C}$. This function specifies that when the cheapest route cost for OD pair $k$ is $\pi_k$, its demand equals $\delta_k(\pi_k)$. Hence, an equilibrium jointly satisfies

Equation (1) and $d_k = \delta_k(\min_{q\in\mathcal{R}_k} c_q(h))$. Depending on the separability of the costs, both the optimization and variational inequality problems can be extended to solve models with elastic demand.

Finally, several authors have looked at other ways to relax some of the basic assumptions. We mention a few examples here. A classical extension to the basic model incorporates capacity and other side-constraints to the equilibrium model [25] as a way of improving the solution quality and correcting the link travel cost functions so as to bring the flow pattern into agreement with the anticipated results. This can be handled by adding the constraints to the formulations and reinterpreting their dual variables as queuing delay caused by congestion. However, this approach is controversial and several authors have looked at alternative models that work better in some situations [26]. Also, Gabriel and Bernstein [27] studied the case of nonadditive models where the cost of a path can be given by more general expressions than just the sum over all arcs in the path. Examples of this include taking into account the variability of travel times and risk-aversion [28], tolls, and valuation of time.

## COMPUTATION OF WARDROP EQUILIBRIA

In this section, we discuss some computational approaches to finding Wardrop equilibria. Let us start by describing the Frank–Wolfe method [29], which is a traditional algorithm that has been used to compute equilibria. It is an iterative descent method that works with the formulation shown in Equation (2) and eventually converges to the equilibrium. The algorithm keeps a current solution, and solves a linearized version of Equation (2) at every step to determine a feasible descent direction. Referring to the objective of that problem by $T(f)$ and to the current solution by $f^i$, the linearized objective is $T(f^i) + \nabla T(f^i)\cdot(f - f^i)$. The linearization enables the algorithm to decompose the problem by OD pairs, allowing it to find a shortest path in $\mathcal{R}_k$ for each commodity $k \in \mathcal{C}$ independently of each other. To identify the steepest descent direction, it computes a shortest path with respect to the prevailing traffic conditions. In the subsequent line search, the original nonlinear problem is solved restricted to the segment defined by the feasible direction of descent. The algorithm terminates when a certain precision is achieved. To determine when this is the case, the convexity of the objective function is used to derive a lower bound on the value of an optimal solution. Alternatively, one can compute the *gap*—defined as the deviation from the shortest path—in the current solution and terminate when it is smaller than a threshold. It is well known that this algorithm always converges to a global minimum because Equation (2) is a convex program.

The standard Frank–Wolfe algorithm sometimes shows poor convergence because it tends to zigzag around the equilibrium solution [8,30–32]. Because "... the (Frank–Wolfe) algorithm is considered sufficiently good for practical use" [30, p. 100], most of the commercial implementations use this procedure. Still, there are many algorithms that were developed to address the slow convergence times. Below, we summarize a few of the main approaches.

LeBlanc *et al*. [33] introduced an improved version of the previous algorithm called *Partan* (parallel tangents), which was further studied by Florian *et al*. [34] and Arezki and Van Vliet [35], among others. This improvement is based on a more intelligent line search. It determines the descent direction using the results of two consecutive iterations, thereby diminishing the zigzagging effect. These two methods belong to a class called *partial linearization algorithms* in which the objective function is simplified to be able to find a search direction.

The structure of Equation (2) leads to *decomposition algorithms*, which separate the main problem into subproblems. The Frank–Wolfe algorithm is an example of this general method since it considers OD pairs separately after fixing the prevailing flows in one iteration. But the separation can be done in other ways, and viewed as a block version of the Gauss–Seidel and Jacobi algorithms. For example, a common decomposition separates flows by node of origin, whereby every iteration assigns all destinations for each origin at the same time. A good example of this approach is given by Bar-Gera's algorithm [36], which is one of the most efficient, in existence, to compute Wardrop equilibria.

The class of *column generation algorithms* deal with a path formulation of the model. Since it is computationally challenging to keep track of the flow along all routes as opposed to maintaining a vector of flows per arc, instead of having one variable per route initially, a column generation algorithm adds routes at the time they are needed. After discovering new routes with the search direction procedure, an algorithm of this type forms a *restricted master problem* that consists of a path formulation of Equation (2) using only the routes discovered thus far (see also ***Column Generation***). These methods are especially important when costs along routes are not additive or when there are constraints based on paths because an arc formulation is not powerful enough to represent the problem in that case. The class of *simplicial decomposition* algorithms finds the next iterate using the restricted master problem. Since all the route information previously computed is badly utilized by algorithms that perform line search, this class can solve problems more efficiently, albeit doing more work per iteration.

The *method of successive averages* is a commonly used heuristic method for computing Wardrop equilibria. This method starts by computing the costs on all arcs for an arbitrary feasible flow. Iteratively, it computes a new solution using an auxiliary linear program that keeps costs fixed, and updates the current solution by averaging it with the new one using a factor that depends on the iteration. This technique is specially useful for more complicated models where exact techniques are not readily available. Some examples are the dynamic and stochastic traffic assignments, see the section titled "More Advanced Models".

The case of elastic demands can be incorporated in the previous discussion since it involves adding another term to the convex minimization problem. The case of nonseparable, symmetric cost functions can be handled similarly to what was described earlier since it admits a convex program formulation. In contrast, the asymmetric case, requires the machinery of variational inequalities or nonlinear complementarity problems. There exist standard algorithms to solve these classes of problems and some of the variants presented earlier for the separable case can be extended to this setting. Notice that since the nonseparable case does not admit a convex programming formulation, checking convergence must rely on regret or other related measures.

The search for efficient algorithms to compute Wardrop equilibria for the various classes of models is a very active area of research. Some of the latest efficient approaches are due to Dial [37], Florian *et al*. [38], Gentile [39], and Bar-Gera [40]. To conclude the discussion on computation, we note that there are some test problems available in the *Transportation Network Test Problems* website [41] that are typically used to study new algorithms.

There exist many commercial software packages that implement some of the algorithms described in this section. These and some additional packages also implement other variants of traffic assignment problems such as dynamic models that explicitly incorporate time [42–44], and simulation models that consider finer behavioral details that analytical models cannot handle [45]. A nonexhaustive list of software implementations is AIMSUN, CUBE, CONTRAM, DYNAMIT, DYNASMART, EMME/2, PARAMICS, TRANSCAD, TRANSIMS, TSIS-CORSIM, SATURN, VISUM-VISSIM, VISTA, and UROAD-UTPS.

## EFFICIENCY OF WARDROP EQUILIBRIA

Since an equilibrium model considers that users unilaterally choose their routes to minimize *their* route cost, the solution is not necessarily efficient. A natural question is thus to quantify how inefficient a Wardrop equilibrium may be, where efficiency is measured as the flow's total travel time $C(f) := \sum_{r\in\mathcal{R}} c_r(h)h_r = \sum_{a\in\mathcal{A}} t_a(f_a)f_a$. Following Wardrop's second principle [5] that states that users minimize the total travel time in the system, a system optimum $f^*$ is an optimal solution to the min-cost multicommodity flow problem:

$$\min\left\{C(f) : f \in X_f\right\}. \qquad (5)$$

It is not hard to observe that, in general, the total travel time incurred by an equilibrium can be arbitrarily larger than that of a social optimum. Consider, for instance, a two-node two-link network with unit demand and cost functions given by $t_1(f_1) = 1$ and $t_2(f_2) = f_2^n$, for some large value of $n$ (Fig. 1). At equilibrium, all flow will use the second link, so that the total travel time will be 1. On the other hand in the system optimum a small fraction of the flow will use the first link, so that its total travel time will be close to 0, making their ratio grow to infinity. Nevertheless, a sequence of results initiated by Roughgarden and Tardos [46] and further developed in papers [47–51] states that if we only allow link travel cost functions belonging to a certain class, then the total travel time of an equilibrium is at most a constant times that of the system optimum.

Let us illustrate this group of results by considering the case in which for all $a \in \mathcal{A}$, $t_a(\cdot)$ is an affine function with nonnegative coefficients. Consider an equilibrium flow $f$ and a system optimal flow $f^*$, then we have that

$$\begin{aligned} C(f) &\le \sum_{a\in\mathcal{A}} t_a(f_a) f_a^* = \sum_{a\in\mathcal{A}} t_a\left(f_a^*\right) f_a^* \\ &\quad + \sum_{a\in\mathcal{A}} f_a^*\left(t_a(f_a) - t_a\left(f_a^*\right)\right) \le \sum_{a\in\mathcal{A}} t_a\left(f_a^*\right) f_a^* \\ &\quad + \sum_{a\in\mathcal{A}: f_a > f_a^*} f_a^*\left(t_a(f_a) - t_a\left(f_a^*\right)\right) \\ &\le \sum_{a\in\mathcal{A}} t_a\left(f_a^*\right) f_a^* + \frac{1}{4}\sum_{a\in\mathcal{A}} t_a(f_a) f_a, \end{aligned}$$

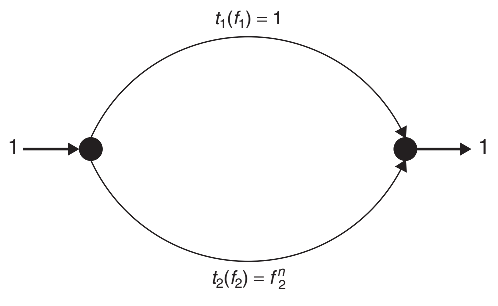


**Figure 1.** Instance where the Wardrop equilibrium is unboundedly worse than the system optimum.

implying that $C(f) \le (4/3) \cdot C(f^*)$. The first inequality in the previous derivation holds because of Equation (4), while the last inequality follows since for affine functions the shaded area in Fig. 2 is at most 25% of the area of the big rectangle. This theorem is due to Roughgarden and Tardos [46], and the short proof presented here appeared in Correa *et al.* [49]. The result implies that, in the worst case among all possible networks, the inefficiency introduced by the self-minded behavior of an equilibrium is never worse than 1/3.

The proof above easily extends to other classes of link travel cost functions by only changing the 25% with the corresponding quantity for a given class of functions. Probably the most interesting aspect of this result is that the efficiency loss depends on the allowable functions rather than on the topology of the network. Furthermore, the idea behind this proof provides another interesting result first derived in Roughgarden and Tardos [46]. Indeed, note that if $t_a(\cdot)$ are arbitrary nondecreasing functions, then $\sum_{a\in\mathcal{A}} x_a t_a(f_a) \le$

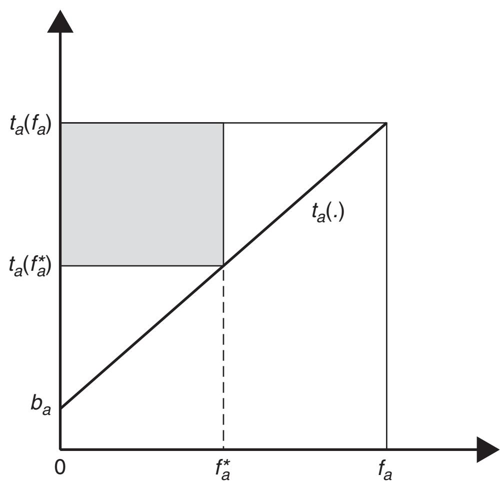


**Figure 2.** Illustration of the proof of the 4/3 result.

$\sum_{a\in\mathcal{A}} \max\{f_a t_a(f_a), x_a t_a(x_a)\} \leq \sum_{a\in\mathcal{A}} f_a t_a(f_a) + \sum_{a\in\mathcal{A}} x_a t_a(x_a)$, where the $f_a$'s and $x_a$'s are any nonnegative numbers. Consider now an equilibrium flow $f$ of a given instance, and a system optimal flow $x$ of a similar instance where demands rates $d$ are doubled. From Equation (4), and since $x/2 \in X_f$, we have

$$C(f) = 2C(f) - C(f) \leq 2\sum_{a\in\mathcal{A}} t_a(f_a)\cdot(x_a/2) - C(f) \leq \sum_{a\in\mathcal{A}} t_a(f_a)\cdot x_a - C(f) \leq C(x).$$

In other words, for arbitrary nondecreasing link travel cost functions, the cost of a Wardrop equilibrium is at most the cost of an optimal solution with the demand doubled. For a restricted set of cost functions, one can provide improved results [49]. For example, under affine costs one can prove that the same statement holds with 25% more demand.

As general equilibria typically do not minimize the social cost, Koutsoupias and Papadimitriou [52] proposed to analyze the inefficiency of equilibria from a worst-case perspective; this led to the notion of "price of anarchy" [53], which is the ratio of the worst social cost of a Nash equilibrium to the cost of an optimal solution. In the context of our traffic model this quantity has been analyzed in a series of papers for increasingly more general classes of cost functions and other model features. The result previously described implies that this worst-case ratio is 4/3. This was extended to more general link travel cost functions by Roughgarden [47] and by Correa *et al*. [48], who basically proved that the efficiency loss in this setting is independent of the topology of the network. Chau and Sim [50] considered the case of nonseparable, symmetric cost functions with elastic demands. They proved that the efficiency loss can be bounded in a similar way as what is described here. Perakis [51] considered general nonseparable cost functions and proved upper bounds using variational inequalities as well. Her bounds depend on two parameters that measure the asymmetry and the nonlinearity of the cost functions considered. Farzad *et al*. [54] provide results of a similar flavor in a closely related model. In their setting the flow (players) have a priority and thus, in any given link a flow particle only experiences a travel time $t_a(x_a)$, where $x_a$ is the amount of flow using link $a$, having higher priority. Interestingly in this context a system optimum is given by Equation (2).

## MORE ADVANCED MODELS

This section discusses some extensions of the basic Wardrop equilibrium model that consider variations on the structure of offer and demand in the network.

In most urban transportation networks, commuters do not have to pay the cost they impose to others by a particular route choice, leading to the bad utilization of the available capacity alluded to in the section titled "Efficiency of Wardrop Equilibria". Since congestion increases sharply with road utilization, having relatively few drivers switch to other routes may significantly improve commute times. Starting with the seminal idea of Vickrey [55,56], many transportation economists have advocated the use of congestion pricing to achieve this goal. The scheme forces drivers to pay a toll when entering congested areas. The underlying idea is to charge drivers the externality they impose to others because when commuters internalize these externalities, the corresponding choices maximize the system welfare. Singapore introduced congestion pricing in 1975, London in 2003 [57,58], and Stockholm in 2007. Increasingly, many large cities have been debating whether a congestion pricing scheme should be adopted. Nevertheless, it has been very hard to implement congestion pricing because of technical, economical, and political problems. There is a large body of research on finding the set of tolls for a given network that optimizes a given objective (e.g., social welfare, revenue, number of tollbooths) under constraints (such as a budget balance, maximum number of tollbooths, or restrictions on their location). For more details, see the books [59,60].

Stochastic user equilibrium models date back to the 1970s, when Dial [61] proposed a model where the demand on each OD is distributed among routes (with random lengths) according to a logit distribution, in the case of uncongested traffic networks. To reduce route enumeration he considered that flow is distributed only among "efficient routes." This model has been widely studied and extended [62,63]. Furthermore, Daganzo and Sheffi [64] looked at the case of dependent route costs, while Fisk [65] studied the model in the context of congested networks, obtaining an equivalent optimization problem. Methods avoiding route enumeration have been proposed by Bell [66], Larsson *et al*. [67], and Maher [68], also leading to equivalent optimization problems in the spirit of Fisk's. Based on the work of Akamatsu [69], Baillon and Cominetti [70] proposed a more general concept called *Markovian traffic equilibrium*, provided an equivalent optimization problem, and established the convergence of the method of successive averages in that context. There are complementary models that consider that travel times themselves are stochastic instead of considering that the perception is stochastic. Some examples are the papers [28,71–75].

Another important extension is modeling public transportation systems. The time commuters need to wait until a bus arrives to the stop adds a difficulty that was not present in models of privately owned vehicles. Indeed, in order to balance waiting and travel time, users' strategies may involve selecting a subset of bus lines and boarding the first available one. This idea was pioneered by Chiriqui and Robillard [76], who considered a network with a single OD pair and $n$ bus lines serving it, each characterized by a travel time and a frequency. They provided a simple efficient algorithm to solve the problem. Spiess and Florian [77] generalized that model to arbitrary networks introducing the notion of general users' strategies. This was further put in graph theoretic terms by Nguyen and Pallotino [78], who called these strategies *hyperpaths* and incorporated travel times dependent on congestion. However, besides increasing travel times, congestion also increases waiting times because a user may not be able to board a selected bus. This is challenging and Wu *et al*. [79] and Bouzaïene-Ayari *et al*. [80] attempted to deal with this problem. Cominetti and Correa [81] considered the frequency of a bus line as a function of the flow in the network and called this function '*effective frequency*.' They proved that an equilibrium in this context exists via an equivalent optimization problem. Cepeda *et al*. [82] obtained a new characterization of equilibria in the congested setting. This led to an effective algorithm that is currently part of EMME/2.

In some situations, such as logistics networks, it is natural to consider that some players control nonnegligible amounts of flow that can be split among several routes, as modeled by an *atomic* network game [83]. In this context, the equilibrium becomes significantly more difficult to characterize and compute because, even for the basic assumptions, the game is generally not potential. Although the existence of equilibria is still guaranteed [84], an instance may possess multiple equilibria [85], and no equivalent convex optimization problem is known even for general separable instances. Furthermore, surprisingly an equilibrium of this game can be less efficient than that of the corresponding nonatomic instance (as in the section titled "The Basic Model"), even in simple networks with two OD pairs [86,87]. Cominetti *et al*. [87] generalized the results of the section titled "Efficiency of Wardrop Equilibria" to this setting, getting a bound of 3/2 when cost functions are afine. It is an open question whether this bound is tight. Swamy [88] proved that tolls that induce an optimal routing of the game with finite players always exist.

Generalizing a classic paper by Rosenthal [89], Milchtaich [90] considered a generalization of the basic model referred to by *congestion games*. In this model the network is abstracted away and each user selects a strategy that consists of a subset of arcs (instead of a route), selected among a set of feasible strategies defined in advance. The defining characteristic of these models is that a link travel cost function for one arc just depends on its demand. Milchtaich himself and others have looked at existence, uniqueness, and computation of equilibria in congestion games with and without atoms, and with homogeneous and heterogeneous cost functions. Actually, some of the results presented in the section titled "Efficiency of Wardrop Equilibria" were originally presented for this kind of model.

Finally, another important area of research involves extending the basic model along the time dimension. Starting with the seminal work of Merchant and Nemhauser [42], many articles have been published about *dynamic user equilibria*. These models consider time-varying conditions on both the offer and demand side of the network. In contrast to the static model discussed here, there is less agreement on the basic characteristics of the model, and on the conditions that such a model should verify, which highlights the difficulty of the problem. Nevertheless, transportation planners commonly make use of tools that rely on analytic or simulation models of this kind. We refer the readers to Ran and Boyce [91], Peeta and Ziliaskopoulos [44], and the report by the Transportation Research Board [92] for more details about dynamic models.

Another relevant issue related to equilibria in general, and to Wardrop equilibria, in particular, is whether players "learn" the equilibria of the game and if so, how fast they do it. There is a rich literature mainly concerned with the dynamics that govern the behavior of players. The book by Sandholm [93] contains an in-depth treatment of the subject. In particular, he describes some learning dynamics under which players of a congestion game, such as those described in this article, converge to an equilibrium. Related to learning and equilibria, there has been limited empirical work to validate whether test subjects behave as the theory of Wardrop equilibria predicts they do [94–98].

## FURTHER READING

For an extensive discussion on modeling and algorithmic techniques, we refer the reader to Magnanti [99], Sheffi [8], Nagurney [100], Patriksson [30], Florian and Hearn [31], and Marcotte and Patriksson [19]. Altman *et al*. [10] surveys the use of this framework in modeling telecommunication networks. Nisan *et al*. [101] and the references therein provide details on the efficiency of equilibria in network games.

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