

A counterexample to the monotonicity conjecture for dynamic equilibria in flows over time

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Abstract

Conjecture 1 of Correa, Cristi and Oosterwijk (*On the price of anarchy for flows over time*, Math. Oper. Res. 47(2), 2022) says that, in the fluid queuing model, sending flow into the network at a higher constant rate can never delay the time at which a fixed amount M of flow has reached the sink. We give a network with seven nodes and eleven edges, rational capacities and delays, for which routing $M = 3/10$ units of flow takes $T_{\text{EQ}} = 72741634397/12569229270 \approx 5.78728$ time units with inflow rate $u_0 = 6$ and $T_{\text{EQ}} = 85124855029/14708672550 \approx 5.78739$ time units with inflow rate $u_0 = 7$. So the higher inflow rate finishes later, and the conjecture is false. Every phase of both equilibria is written out, so the computation can be checked by hand with the equilibrium conditions of the paper.

1 The model and the conjecture

We use exactly the notation of the paper. A network $G = (V, E)$ has a source s , a sink t , and every edge e has a capacity $\nu_e > 0$ and a delay $\tau_e \geq 0$. An amount M of flow has to travel from s to t ; it leaves s at the constant rate u_0 during the time interval $[0, M/u_0]$.

Edges. If flow enters edge e at rate $f_e^+(\theta)$ above the capacity ν_e , a queue builds up at the tail of e ; $z_e(\theta)$ is the amount of flow in that queue at time θ , and it changes at rate $f_e^+(\theta) - \nu_e$ while $z_e(\theta) > 0$. A particle entering e at time θ waits $z_e(\theta)/\nu_e$ in the queue and then travels for τ_e , so it leaves e at time

$$T_e(\theta) = \theta + \frac{z_e(\theta)}{\nu_e} + \tau_e.$$

Labels. For the particle that leaves s at time θ , $\ell_v(\theta)$ is the earliest time at which it can be at node v :

$$\ell_s(\theta) = \theta, \quad \ell_v(\theta) = \min_{e=(u,v) \in E} T_e(\ell_u(\theta)) \quad (v \neq s).$$

In a *dynamic equilibrium* every particle travels along a fastest route, that is, flow enters an edge $e = (v, w)$ at time $\ell_v(\theta)$ only if e is *active* for θ ,

$$E'_\theta = \{e = (v, w) \in E : \ell_w(\theta) \geq \ell_v(\theta) + \tau_e\}.$$

The edges on which the particle θ finds a queue are

$$E_\theta^* = \{e = (v, w) \in E : \ell_w(\theta) > \ell_v(\theta) + \tau_e\} \subseteq E'_\theta,$$

*The author's contribution was minimal. The only reason he is an author is that today's standards require a human to be an author. The counterexample was obtained by ChatGPT Astra and checked by Claude Fable 5.1. The write up is also AI generated, but the author finds it clear enough to be red by humans.

and for $e \in E_\theta^*$ the size of that queue when the particle reaches the tail of e is $z_e(\ell_v(\theta)) = \nu_e(\ell_w(\theta) - \ell_v(\theta) - \tau_e)$; edges not in E_θ^* have no queue at that moment.

Makespan. Particles arrive at t in the order in which they left s , so the last particle, which leaves s at time M/u_0 , is the last to arrive, at time

$$T_{\text{EQ}} = \ell_t(M/u_0).$$

This is the makespan of the equilibrium for routing M units with inflow rate u_0 .

Conjecture 1 (Correa, Cristi, Oosterwijk). *Consider a network G and two fixed inflow rates $u_1 < u_2$ with their corresponding dynamic equilibria in G and their corresponding makespans T_{EQ}^1 and T_{EQ}^2 for routing M units of flow. Then $T_{\text{EQ}}^1 \geq T_{\text{EQ}}^2$.*

Theorem 1. *For the network of Section 2 and $M = 3/10$, the equilibrium makespans are*

$$\begin{aligned} T_{\text{EQ}}^1 &= \ell_t\left(\frac{1}{20}\right) = \frac{72741634397}{12569229270} \approx 5.787278825 && \text{for } u_1 = 6, \\ T_{\text{EQ}}^2 &= \ell_t\left(\frac{3}{70}\right) = \frac{85124855029}{14708672550} \approx 5.787392080 && \text{for } u_2 = 7. \end{aligned}$$

Hence $T_{\text{EQ}}^1 < T_{\text{EQ}}^2$ although $u_1 < u_2$; the difference is $T_{\text{EQ}}^2 - T_{\text{EQ}}^1 = 334592/2954306025 \approx 1.13 \cdot 10^{-4}$. Conjecture 1 is false.

The rest of the note is the computation of the two equilibria, phase by phase.

2 The network

The nodes are $s = v_0, v_1, \dots, v_5, t = v_6$ and the edges $e_0, \dots, e_{10}$ are given in Figure 1 and Table 1. Edges e_2 and e_3 are two different edges from v_1 to v_2 (a slow one with no delay and a fast one with delay 2). The path $s \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_5 \rightarrow t$ along $e_0, e_2, e_4, e_6, e_9, e_{10}$ has total delay 0; its capacities decrease along the path $(5, \frac{1}{8}, \frac{1}{20}, \frac{1}{25})$, then increase again $(\frac{1}{8}, \frac{1}{16})$.

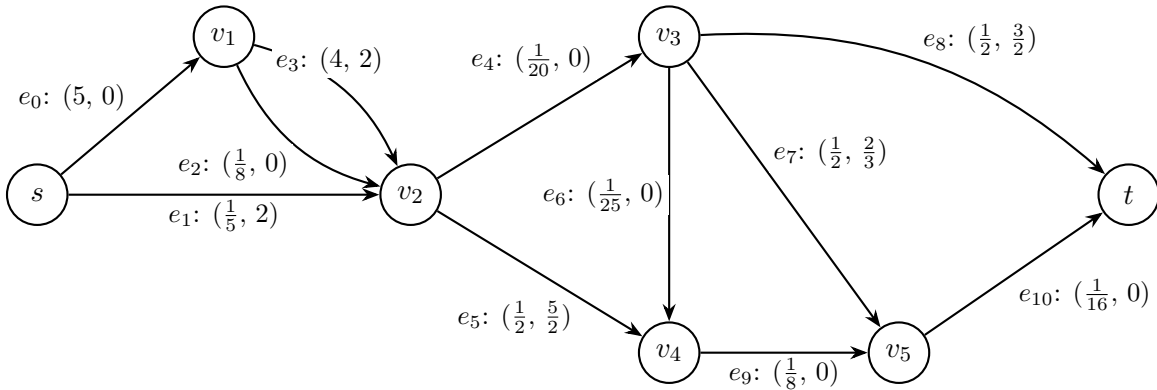


Figure 1: The network. Each edge is labelled $e_i: (\nu_{e_i}, \tau_{e_i}) = (\text{capacity}, \text{delay})$.

edge	from $\rightarrow$ to	ν_e	τ_e	edge	from $\rightarrow$ to	ν_e	τ_e
e_0	$s \rightarrow v_1$	5	0	e_6	$v_3 \rightarrow v_4$	1/25	0
e_1	$s \rightarrow v_2$	1/5	2	e_7	$v_3 \rightarrow v_5$	1/2	2/3
e_2	$v_1 \rightarrow v_2$	1/8	0	e_8	$v_3 \rightarrow t$	1/2	3/2
e_3	$v_1 \rightarrow v_2$	4	2	e_9	$v_4 \rightarrow v_5$	1/8	0
e_4	$v_2 \rightarrow v_3$	1/20	0	e_{10}	$v_5 \rightarrow t$	1/16	0
e_5	$v_2 \rightarrow v_4$	1/2	5/2				

Table 1: Capacities and delays.

3 How the equilibrium is computed, and how to read the tables

We use Theorem 1 of the paper and the phase-by-phase construction described after it. The equilibrium consists of consecutive *phases* $[\theta_0, \theta_1), [\theta_1, \theta_2), \dots$ with $\theta_0 = 0$. Inside a phase the sets E'_θ and E_θ^* do not change, the labels are affine,

$$\ell_v(\theta) = \ell_v(\theta_j) + (\theta - \theta_j) \ell'_v \quad (\theta_j \leq \theta \leq \theta_{j+1}),$$

and the derivatives ℓ'_v together with the derivatives x'_e of the cumulative edge flows (the paper's normalized thin flow with resetting) are constant and satisfy:

- (a) $\ell'_s = 1$, and $x' = (x'_e)_{e \in E}$ is a static s - t flow of value u_0 that uses only active edges ($x'_e = 0$ for $e \notin E'_\theta$);
- (b) for every active edge $e = (v, w)$ the paper's conditions (5)–(7):

$$\ell'_w \leq \ell'_v \quad \text{if } e \in E'_\theta \setminus E_\theta^* \text{ and } x'_e = 0, \quad (5)$$

$$\ell'_w = \max\{\ell'_v, x'_e/\nu_e\} \quad \text{if } e \in E'_\theta \setminus E_\theta^* \text{ and } x'_e > 0, \quad (6)$$

$$\ell'_w = x'_e/\nu_e \quad \text{if } e \in E_\theta^*; \quad (7)$$

- (c) for every $w \neq s$, ℓ'_w equals the smallest of the right-hand sides in (5)–(7) over the active edges $e = (v, w)$ entering w (this is the derivative of $\ell_w(\theta) = \min_e T_e(\ell_v(\theta))$).

The physical meaning is simple: the flow entering $e = (v, w)$ at time $\ell_v(\theta)$ has rate x'_e/ℓ'_v ; if e has a queue, flow leaves e at rate ν_e , so the particles entering e during $[\ell_v(\theta), \ell_v(\theta + d\theta)]$ leave during an interval of length $x'_e d\theta/\nu_e$, which is (7); if e has no queue and $x'_e/\ell'_v \leq \nu_e$, they leave at the rate they entered, which is the first case of (6); if $x'_e/\ell'_v > \nu_e$ a queue starts to form, which is the second case of (6).

A phase ends at the first time θ_{j+1} at which either an inactive edge becomes active (the difference $\ell_w(\theta) - \ell_v(\theta)$ grows to τ_e) or a queue empties ($\ell_w(\theta) - \ell_v(\theta) - \tau_e$ shrinks to 0); then E'_θ, E_θ^* change and new derivatives are computed. The labels ℓ of the dynamic equilibrium are unique (Cominetti, Correa and Larré), so there is nothing to choose in this procedure. In our network no queue ever empties before the last particle: all five events are activations of new edges.

For each inflow rate we give five tables: the phases with E'_θ, E_θ^* and the event that ends each phase (Tables 2, 7); the derivatives ℓ'_v (Tables 3, 8); the flow derivatives x'_e (Tables 4, 9); the labels $\ell_v(\theta_j)$ at the start of every phase and for the last particle (Tables 5, 10); and the queue sizes $z_e(\ell_v(\theta_j))$ met by the particle θ_j (Tables 6, 11). Each row of the label tables is obtained from the previous one by adding $(\theta_{j+1} - \theta_j)\ell'_j$; the makespan is the last entry of the last row. Every number is an exact rational.

4 Inflow rate $u_0 = 7$

The flow leaves s during $[0, \frac{3}{70}]$. The six phases are the following.

Phase 0, $\theta \in [0, \frac{2}{105})$. All flow takes the zero-delay path $s, v_1, v_2, v_3, v_4, v_5, t$. Edge e_0 receives $7 > 5 = \nu_{e_0}$, so a queue forms on it immediately, and e_0 delivers flow to v_1 at rate 5; likewise e_2 receives $5 > \frac{1}{8}$, e_4 receives $\frac{1}{8} > \frac{1}{20}$ and e_6 receives $\frac{1}{20} > \frac{1}{25}$, so queues form on e_2, e_4, e_6 as well. Edges e_9 and e_{10} receive $\frac{1}{25}$, below their capacities, and have no queue. Thus $\ell'_{v_1} = x'_{e_0}/\nu_{e_0} = 7/5$, $\ell'_{v_2} = 7 \cdot 8 = 56$, $\ell'_{v_3} = 7 \cdot 20 = 140$, $\ell'_{v_4} = 7 \cdot 25 = 175$, and $\ell'_{v_5} = \ell'_t = 175$ by (6) with the first case of the maximum. The sink receives flow at rate $u_0/\ell'_t = 1/25$. The phase ends when $\ell_{v_5}(\theta) - \ell_{v_3}(\theta) = 35\theta$ reaches $\tau_{e_7} = \frac{2}{3}$, at $\theta_1 = \frac{2}{105}$: the detour e_7 from v_3 to v_5 becomes as fast as waiting in the queue of e_6 .

Phase 1, $\theta \in [\frac{2}{105}, \frac{11}{504})$. At v_3 the flow splits between e_6 (still with a queue) and e_7 (no queue): $x'_{e_6} = \frac{28}{5}$, $x'_{e_7} = \frac{7}{5}$. Now $\ell'_{v_4} = \ell'_{v_5} = \ell'_t = 140$ and the sink receives flow at rate $1/20$. The phase ends when $\ell_{v_4} - \ell_{v_2}$ reaches $\tau_{e_5} = \frac{5}{2}$: edge e_5 becomes active.

Phase 2, $\theta \in [\frac{11}{504}, \frac{2}{55})$. At v_2 the flow splits between e_4 and e_5 ; $\ell'_{v_3} = \ell'_{v_4} = \ell'_{v_5} = 56$. Edge e_{10} now receives flow at rate $x'_{e_{10}}/\ell'_{v_5} = 7/56 = \frac{1}{8} > \frac{1}{16}$, so a queue forms on e_{10} and $\ell'_t = 7 \cdot 16 = 112$; the sink receives flow at rate $1/16$. The phase ends when $\ell_{v_2} - \ell_s$ reaches $\tau_{e_1} = 2$: the slow direct edge e_1 from s to v_2 becomes active.

Phase 3, $\theta \in [\frac{2}{55}, \frac{31639}{863280})$. At s the flow splits, $x'_{e_0} = \frac{35}{13}$ and $x'_{e_1} = \frac{56}{13}$; e_1 receives $x'_{e_1}/\ell'_s = \frac{56}{13} > \frac{1}{5}$ and gets a queue, and $\ell'_{v_2} = x'_{e_1}/\nu_{e_1} = \frac{280}{13}$. Edge e_0 now receives only $\frac{35}{13} < 5$, so its queue shrinks (it stays positive until the end), and $\ell'_{v_1} = \frac{7}{13}$. Edge e_9 gets a queue. The sink still receives flow at rate $1/16$. The phase ends when $\ell_t - \ell_{v_3}$ reaches $\tau_{e_8} = \frac{3}{2}$: the direct edge e_8 from v_3 to t becomes active.

Phase 4, $\theta \in [\frac{31639}{863280}, \frac{214}{5775})$. At v_3 the flow now splits between e_6 and e_8 , and e_7 carries no flow: at θ_4 we have $\ell'_{v_5} = \frac{4088}{117} < \frac{8176}{117} = \ell'_{v_3}$, so (5) holds strictly and e_7 becomes inactive right after θ_4 . The sink receives flow through e_8 (rate $x'_{e_8}/\ell'_{v_3} = \frac{308/117}{8176/117} = \frac{11}{292}$) and e_{10} (rate $\nu_{e_{10}} = \frac{1}{16}$), in total $u_0/\ell'_t = \frac{117}{1168}$. The phase ends when $\ell_{v_2} - \ell_{v_1}$ reaches $\tau_{e_3} = 2$: the fast edge e_3 from v_1 to v_2 becomes active.

Phase 5, $\theta \in [\frac{214}{5775}, \frac{3}{70}]$. Almost all the flow arriving at v_1 now uses e_3 ($x'_{e_3} = \frac{1120}{173}$, $x'_{e_2} = \frac{35}{173}$), a queue forms on e_3 , and e_5 gets a queue as well. The sink receives flow at rate $u_0/\ell'_t = \frac{233}{2160}$. The last particle leaves s at $\theta = M/u_0 = \frac{3}{70}$ and arrives at t at time

$$T_{\text{EQ}}^2 = \ell_t\left(\frac{3}{70}\right) = \frac{85124855029}{14708672550} \approx 5.787392080.$$

Table 2: $u_0 = 7$: the phases. The sets are given by the indices i of the edges e_i ; they are constant on the open interval (θ_j, θ_{j+1}) .

j	$[\theta_j, \theta_{j+1})$	E'_θ (active)	E_θ^* (queue)	event at θ_{j+1}
0	$[0, \frac{2}{105})$	0, 2, 4, 6, 9, 10	0, 2, 4, 6	e_7 becomes active: $\ell_{v_5} - \ell_{v_3} = \tau_{e_7} = \frac{2}{3}$
1	$[\frac{2}{105}, \frac{11}{504})$	0, 2, 4, 6, 7, 9, 10	0, 2, 4, 6	e_5 becomes active: $\ell_{v_4} - \ell_{v_2} = \tau_{e_5} = \frac{5}{2}$
2	$[\frac{11}{504}, \frac{2}{55})$	0, 2, 4, 5, 6, 7, 9, 10	0, 2, 4, 6, 10	e_1 becomes active: $\ell_{v_2} - \ell_s = \tau_{e_1} = 2$
3	$[\frac{2}{55}, \frac{31639}{863280})$	0, 1, 2, 4, 5, 6, 7, 9, 10	0, 1, 2, 4, 6, 9, 10	e_8 becomes active: $\ell_t - \ell_{v_3} = \tau_{e_8} = \frac{3}{2}$
4	$[\frac{31639}{863280}, \frac{214}{5775})$	0, 1, 2, 4, 5, 6, 8, 9, 10	0, 1, 2, 4, 6, 9, 10	e_3 becomes active: $\ell_{v_2} - \ell_{v_1} = \tau_{e_3} = 2$
5	$[\frac{214}{5775}, \frac{3}{70})$	0, 1, 2, 3, 4, 5, 6, 8, 9, 10	0, 1, 2, 3, 4, 5, 6, 9, 10	last particle ($\theta = M/u_0$)

Table 3: $u_0 = 7$: label derivatives $\ell'_v(\theta)$ on each phase (always $\ell'_s = 1$).

j	ℓ'_s	ℓ'_{v_1}	ℓ'_{v_2}	ℓ'_{v_3}	ℓ'_{v_4}	ℓ'_{v_5}	ℓ'_t
0	1	$\frac{7}{5}$	56	140	175	175	175
1	1	$\frac{7}{5}$	56	140	140	140	140
2	1	$\frac{7}{5}$	56	56	56	56	112
3	1	$\frac{7}{13}$	$\frac{280}{13}$	$\frac{584}{13}$	$\frac{280}{13}$	$\frac{584}{13}$	112
4	1	$\frac{7}{13}$	$\frac{280}{13}$	$\frac{8176}{117}$	$\frac{280}{13}$	$\frac{4088}{117}$	$\frac{8176}{117}$
5	1	$\frac{231}{173}$	$\frac{280}{173}$	$\frac{15120}{233}$	$\frac{1750}{233}$	$\frac{7560}{233}$	$\frac{15120}{233}$

Table 4: $u_0 = 7$: the thin flow $x'_e(\theta)$ on each phase (a static s - t flow of value $u_0 = 7$).

j	x'_{e_0}	x'_{e_1}	x'_{e_2}	x'_{e_3}	x'_{e_4}	x'_{e_5}	x'_{e_6}	x'_{e_7}	x'_{e_8}	x'_{e_9}	$x'_{e_{10}}$
0	7	0	7	0	7	0	7	0	0	7	7
1	7	0	7	0	7	0	$\frac{28}{5}$	$\frac{7}{5}$	0	$\frac{28}{5}$	7
2	7	0	7	0	$\frac{14}{5}$	$\frac{21}{5}$	$\frac{56}{25}$	$\frac{14}{25}$	0	$\frac{161}{25}$	7
3	$\frac{35}{13}$	$\frac{56}{13}$	$\frac{35}{13}$	0	$\frac{146}{65}$	$\frac{309}{65}$	$\frac{56}{65}$	$\frac{18}{13}$	0	$\frac{73}{13}$	7
4	$\frac{35}{13}$	$\frac{56}{13}$	$\frac{35}{13}$	0	$\frac{2044}{585}$	$\frac{2051}{585}$	$\frac{56}{65}$	0	$\frac{308}{117}$	$\frac{511}{117}$	$\frac{511}{117}$
5	$\frac{1155}{173}$	$\frac{56}{173}$	$\frac{35}{173}$	$\frac{1120}{173}$	$\frac{756}{233}$	$\frac{875}{233}$	$\frac{70}{233}$	0	$\frac{686}{233}$	$\frac{945}{233}$	$\frac{945}{233}$

Table 5: $u_0 = 7$: labels $\ell_v(\theta)$ at the phase starts and for the last particle $\theta = M/u_0 = \frac{3}{70}$. The last column is the arrival time at t .

θ	$\ell_s(\theta)$	$\ell_{v_1}(\theta)$	$\ell_{v_2}(\theta)$	$\ell_{v_3}(\theta)$	$\ell_{v_4}(\theta)$	$\ell_{v_5}(\theta)$	$\ell_t(\theta)$
$\theta_0 = 0$	0	0	0	0	0	0	0
$\theta_1 = \frac{2}{105}$	$\frac{2}{105}$	$\frac{2}{75}$	$\frac{16}{15}$	$\frac{8}{3}$	$\frac{10}{3}$	$\frac{10}{3}$	$\frac{10}{3}$
$\theta_2 = \frac{11}{504}$	$\frac{11}{504}$	$\frac{11}{360}$	$\frac{11}{9}$	$\frac{55}{18}$	$\frac{67}{18}$	$\frac{67}{18}$	$\frac{67}{18}$
$\theta_3 = \frac{2}{55}$	$\frac{2}{55}$	$\frac{14}{275}$	$\frac{112}{55}$	$\frac{1277}{330}$	$\frac{499}{110}$	$\frac{499}{110}$	$\frac{5297}{990}$
$\theta_4 = \frac{31639}{863280}$	$\frac{31639}{863280}$	$\frac{220409}{4316400}$	$\frac{220409}{107910}$	$\frac{209483}{53955}$	$\frac{245092}{53955}$	$\frac{245453}{53955}$	$\frac{580831}{107910}$
$\theta_5 = \frac{214}{5775}$	$\frac{214}{5775}$	$\frac{2}{39}$	$\frac{80}{39}$	$\frac{123444214}{31563675}$	$\frac{355}{78}$	$\frac{288076669}{63127350}$	$\frac{341579453}{63127350}$
$M/u_0 = \frac{3}{70}$	$\frac{3}{70}$	$\frac{19913}{337350}$	$\frac{764684}{371085}$	$\frac{31530923102}{7354336275}$	$\frac{918575}{199914}$	$\frac{69890285117}{14708672550}$	$\frac{85124855029}{14708672550}$

Table 6: $u_0 = 7$: queue mass $z_e(\ell_v(\theta)) = \nu_e(\ell_w(\theta) - \ell_v(\theta) - \tau_e)^+$ that the particle θ finds on $e = (v, w)$ when it reaches v (zero for edges not shown).

θ	z_{e_0}	z_{e_1}	z_{e_2}	z_{e_3}	z_{e_4}	z_{e_5}	z_{e_6}	z_{e_9}	$z_{e_{10}}$
θ_0	0	0	0	0	0	0	0	0	0
θ_1	$\frac{4}{105}$	0	$\frac{13}{100}$	0	$\frac{2}{25}$	0	$\frac{2}{75}$	0	0
θ_2	$\frac{11}{252}$	0	$\frac{143}{960}$	0	$\frac{11}{120}$	0	$\frac{2}{75}$	0	0
θ_3	$\frac{4}{55}$	0	$\frac{273}{1100}$	0	$\frac{11}{120}$	0	$\frac{2}{75}$	0	$\frac{403}{7920}$
θ_4	$\frac{10369}{143880}$	$\frac{1691}{1438800}$	$\frac{2865317}{11510400}$	0	$\frac{198557}{2158200}$	0	$\frac{35609}{1348875}$	$\frac{361}{431640}$	$\frac{5}{96}$
θ_5	$\frac{356}{5005}$	$\frac{356}{125125}$	$\frac{1}{4}$	0	$\frac{29349107}{315636750}$	0	$\frac{40421947}{1578183750}$	$\frac{383147}{252509400}$	$\frac{1671962}{31563675}$
M/u_0	$\frac{19093}{236145}$	$\frac{18509}{5195190}$	$\frac{571369}{2283600}$	$\frac{938}{142725}$	$\frac{1416528978833}{12723001755750}$	$\frac{75777}{4433990}$	$\frac{4522309421}{367716813750}$	$\frac{576532373}{29417345100}$	$\frac{1904321239}{29417345100}$

5 Inflow rate $u_0 = 6$

The flow leaves s during $[0, \frac{1}{20}]$. The sequence of events is the same as for $u_0 = 7$ (the same five edges become active in the same order, and no queue empties), only the times and the numbers change; in particular all ℓ'_v for $v \neq s$ and all x'_e are $6/7$ of the corresponding values for $u_0 = 7$, while the phase boundaries θ_j are *not* simply rescaled. The last particle leaves s at $\theta = \frac{1}{20}$ and arrives at t at time

$$T_{\text{EQ}}^1 = \ell_t\left(\frac{1}{20}\right) = \frac{72741634397}{12569229270} \approx 5.787278825.$$

Table 7: $u_0 = 6$: the phases. The sets are given by the indices i of the edges e_i ; they are constant on the open interval (θ_j, θ_{j+1}) .

j	$[\theta_j, \theta_{j+1})$	E'_θ (active)	E_θ^* (queue)	event at θ_{j+1}
0	$[0, \frac{1}{45})$	0, 2, 4, 6, 9, 10	0, 2, 4, 6	e_7 becomes active: $\ell_{v_5} - \ell_{v_3} = \tau_{e_7} = \frac{2}{3}$
1	$[\frac{1}{45}, \frac{11}{432})$	0, 2, 4, 6, 7, 9, 10	0, 2, 4, 6	e_5 becomes active: $\ell_{v_4} - \ell_{v_2} = \tau_{e_5} = \frac{5}{2}$
2	$[\frac{11}{432}, \frac{2}{47})$	0, 2, 4, 5, 6, 7, 9, 10	0, 2, 4, 6, 10	e_1 becomes active: $\ell_{v_2} - \ell_s = \tau_{e_1} = 2$
3	$[\frac{2}{47}, \frac{189353}{4426272})$	0, 1, 2, 4, 5, 6, 7, 9, 10	0, 1, 2, 4, 6, 9, 10	e_8 becomes active: $\ell_t - \ell_{v_3} = \tau_{e_8} = \frac{3}{2}$
4	$[\frac{189353}{4426272}, \frac{91}{2115})$	0, 1, 2, 4, 5, 6, 8, 9, 10	0, 1, 2, 4, 6, 9, 10	e_3 becomes active: $\ell_{v_2} - \ell_{v_1} = \tau_{e_3} = 2$
5	$[\frac{91}{2115}, \frac{1}{20})$	0, 1, 2, 3, 4, 5, 6, 8, 9, 10	0, 1, 2, 3, 4, 5, 6, 9, 10	last particle ($\theta = M/u_0$)

Table 8: $u_0 = 6$: label derivatives $\ell'_v(\theta)$ on each phase (always $\ell'_s = 1$).

j	ℓ'_s	ℓ'_{v_1}	ℓ'_{v_2}	ℓ'_{v_3}	ℓ'_{v_4}	ℓ'_{v_5}	ℓ'_t
0	1	$\frac{6}{5}$	48	120	150	150	150
1	1	$\frac{6}{5}$	48	120	120	120	120
2	1	$\frac{6}{5}$	48	48	48	48	96
3	1	$\frac{6}{13}$	$\frac{240}{13}$	$\frac{3504}{91}$	$\frac{240}{13}$	$\frac{3504}{91}$	96
4	1	$\frac{6}{13}$	$\frac{240}{13}$	$\frac{2336}{39}$	$\frac{240}{13}$	$\frac{1168}{39}$	$\frac{2336}{39}$
5	1	$\frac{198}{173}$	$\frac{240}{173}$	$\frac{12960}{233}$	$\frac{1500}{233}$	$\frac{6480}{233}$	$\frac{12960}{233}$

 Table 9: $u_0 = 6$: the thin flow $x'_e(\theta)$ on each phase (a static s - t flow of value $u_0 = 6$).

j	x'_{e_0}	x'_{e_1}	x'_{e_2}	x'_{e_3}	x'_{e_4}	x'_{e_5}	x'_{e_6}	x'_{e_7}	x'_{e_8}	x'_{e_9}	$x'_{e_{10}}$
0	6	0	6	0	6	0	6	0	0	6	6
1	6	0	6	0	6	0	$\frac{24}{5}$	$\frac{6}{5}$	0	$\frac{24}{5}$	6
2	6	0	6	0	$\frac{12}{5}$	$\frac{18}{5}$	$\frac{48}{25}$	$\frac{12}{25}$	0	$\frac{138}{25}$	6
3	$\frac{30}{13}$	$\frac{48}{13}$	$\frac{30}{13}$	0	$\frac{876}{455}$	$\frac{1854}{455}$	$\frac{48}{65}$	$\frac{108}{91}$	0	$\frac{438}{91}$	6
4	$\frac{30}{13}$	$\frac{48}{13}$	$\frac{30}{13}$	0	$\frac{584}{195}$	$\frac{586}{195}$	$\frac{48}{65}$	0	$\frac{88}{39}$	$\frac{146}{39}$	$\frac{146}{39}$
5	$\frac{990}{173}$	$\frac{48}{173}$	$\frac{30}{173}$	$\frac{960}{173}$	$\frac{648}{233}$	$\frac{750}{233}$	$\frac{60}{233}$	0	$\frac{588}{233}$	$\frac{810}{233}$	$\frac{810}{233}$

 Table 10: $u_0 = 6$: labels $\ell_v(\theta)$ at the phase starts and for the last particle $\theta = M/u_0 = \frac{1}{20}$. The last column is the arrival time at t .

θ	$\ell_s(\theta)$	$\ell_{v_1}(\theta)$	$\ell_{v_2}(\theta)$	$\ell_{v_3}(\theta)$	$\ell_{v_4}(\theta)$	$\ell_{v_5}(\theta)$	$\ell_t(\theta)$
$\theta_0 = 0$	0	0	0	0	0	0	0
$\theta_1 = \frac{1}{45}$	$\frac{1}{45}$	$\frac{2}{75}$	$\frac{16}{15}$	$\frac{8}{3}$	$\frac{10}{3}$	$\frac{10}{3}$	$\frac{10}{3}$
$\theta_2 = \frac{11}{432}$	$\frac{11}{432}$	$\frac{11}{360}$	$\frac{11}{9}$	$\frac{55}{18}$	$\frac{67}{18}$	$\frac{67}{18}$	$\frac{67}{18}$
$\theta_3 = \frac{2}{47}$	$\frac{2}{47}$	$\frac{12}{235}$	$\frac{96}{47}$	$\frac{1093}{282}$	$\frac{427}{94}$	$\frac{427}{94}$	$\frac{4537}{846}$
$\theta_4 = \frac{189353}{4426272}$	$\frac{189353}{4426272}$	$\frac{188737}{3688560}$	$\frac{188737}{92214}$	$\frac{179107}{46107}$	$\frac{209636}{46107}$	$\frac{209845}{46107}$	$\frac{496535}{92214}$
$\theta_5 = \frac{91}{2115}$	$\frac{91}{2115}$	$\frac{2}{39}$	$\frac{80}{39}$	$\frac{105176102}{26972595}$	$\frac{355}{78}$	$\frac{245917157}{53945190}$	$\frac{291269989}{53945190}$
$M/u_0 = \frac{1}{20}$	$\frac{1}{20}$	$\frac{187931}{3171090}$	$\frac{653548}{317109}$	$\frac{26943895246}{6284614635}$	$\frac{3925955}{854178}$	$\frac{59736561061}{12569229270}$	$\frac{72741634397}{12569229270}$

Table 11: $u_0 = 6$: queue mass $z_e(\ell_v(\theta)) = \nu_e(\ell_w(\theta) - \ell_v(\theta) - \tau_e)^+$ that the particle θ finds on $e = (v, w)$ when it reaches v (zero for edges not shown).

θ	z_{e_0}	z_{e_1}	z_{e_2}	z_{e_3}	z_{e_4}	z_{e_5}	z_{e_6}	z_{e_9}	$z_{e_{10}}$
θ_0	0	0	0	0	0	0	0	0	0
θ_1	$\frac{1}{45}$	0	$\frac{13}{100}$	0	$\frac{2}{25}$	0	$\frac{2}{75}$	0	0
θ_2	$\frac{11}{432}$	0	$\frac{143}{960}$	0	$\frac{11}{120}$	0	$\frac{2}{75}$	0	0
θ_3	$\frac{2}{47}$	0	$\frac{117}{470}$	0	$\frac{11}{120}$	0	$\frac{2}{75}$	0	$\frac{347}{6768}$
θ_4	$\frac{185657}{4426272}$	$\frac{17479}{22131360}$	$\frac{2453581}{9836160}$	0	$\frac{169477}{1844280}$	0	$\frac{30529}{1152675}$	$\frac{209}{368856}$	$\frac{5}{96}$
θ_5	$\frac{227}{5499}$	$\frac{137475}{137475}$	$\frac{1}{4}$	0	$\frac{24923851}{269725950}$	0	$\frac{35167571}{1348629750}$	$\frac{198691}{215780760}$	$\frac{1417276}{26972595}$
M/u_0	$\frac{58753}{1268436}$	$\frac{69491}{31710900}$	$\frac{488273}{1951440}$	$\frac{826}{121965}$	$\frac{1210271236249}{10872383318550}$	$\frac{66729}{3789046}$	$\frac{3882637333}{314230731750}$	$\frac{491533309}{25138458540}$	$\frac{34587961}{534860820}$

6 Checking a phase by hand

Any phase can be checked from the tables with conditions (a)–(c) of Section 3. We do it for phase 3 of $u_0 = 7$ ($\theta \in [\frac{2}{55}, \frac{31639}{863280})$), where

$$\ell' = (\ell'_s, \ell'_{v_1}, \ell'_{v_2}, \ell'_{v_3}, \ell'_{v_4}, \ell'_{v_5}, \ell'_t) = \left(1, \frac{7}{13}, \frac{280}{13}, \frac{584}{13}, \frac{280}{13}, \frac{584}{13}, 112\right)$$

and

$$x' = (x'_{e_0}, \dots, x'_{e_{10}}) = \left(\frac{35}{13}, \frac{56}{13}, \frac{35}{13}, 0, \frac{146}{65}, \frac{309}{65}, \frac{56}{65}, \frac{18}{13}, 0, \frac{73}{13}, 7\right).$$

Active edges. From Table 5, at $\theta_3 = \frac{2}{55}$ the labels are $\ell = (\frac{2}{55}, \frac{14}{275}, \frac{112}{55}, \frac{1277}{330}, \frac{499}{110}, \frac{499}{110}, \frac{5297}{990})$. The differences $\ell_w - \ell_v - \tau_e$ are positive for $e_0, e_2, e_4, e_6, e_9, e_{10}$ (queues), zero for e_1 (it has just become active; it gets a queue immediately because $x'_{e_1}/\nu_{e_1} = \frac{280}{13} > \ell'_s = 1$), zero for e_5 and e_7 (active, no queue, and $\ell'_{v_4} = \ell'_{v_2}$, $\ell'_{v_5} = \ell'_{v_3}$ keep them so), and negative for e_3 ($\frac{112}{55} - \frac{14}{275} - 2 = -\frac{4}{275}$) and e_8 ($\frac{5297}{990} - \frac{1277}{330} - \frac{3}{2} = -\frac{19}{990}$), which are inactive and carry no flow.

Flow conservation (a). At s : $\frac{35}{13} + \frac{56}{13} = 7 = u_0$. At v_1 : in $\frac{35}{13}$, out $\frac{35}{13} + 0$. At v_2 : in $\frac{56}{13} + \frac{35}{13} = 7$, out $\frac{146}{65} + \frac{309}{65} = 7$. At v_3 : in $\frac{146}{65}$, out $\frac{56}{65} + \frac{18}{13} + 0 = \frac{146}{65}$. At v_4 : in $\frac{309}{65} + \frac{56}{65} = \frac{73}{13}$, out $\frac{73}{13}$. At v_5 : in $\frac{18}{13} + \frac{73}{13} = 7$, out 7. At t : in $0 + 7 = 7$.

Conditions (5)–(7) and (c). Edges with a queue, condition (7): e_0 : $\ell'_{v_1} = \frac{35/13}{5} = \frac{7}{13}$; e_1 : $\ell'_{v_2} = \frac{56/13}{1/5} = \frac{280}{13}$; e_2 : $\ell'_{v_2} = \frac{35/13}{1/8} = \frac{280}{13}$ (consistent with e_1); e_4 : $\ell'_{v_3} = \frac{146/65}{1/20} = \frac{584}{13}$; e_6 : $\ell'_{v_4} = \frac{56/65}{1/25} = \frac{280}{13}$; e_9 : $\ell'_{v_5} = \frac{73/13}{1/8} = \frac{584}{13}$; e_{10} : $\ell'_t = \frac{7}{1/16} = 112$. Active edges without queue carrying flow, condition (6): e_5 : $\max\{\ell'_{v_2}, x'_{e_5}/\nu_{e_5}\} = \max\{\frac{280}{13}, \frac{618}{65}\} = \frac{280}{13} = \ell'_{v_4}$ (consistent with e_6); e_7 : $\max\{\ell'_{v_3}, x'_{e_7}/\nu_{e_7}\} = \max\{\frac{584}{13}, \frac{36}{13}\} = \frac{584}{13} = \ell'_{v_5}$ (consistent with e_9). There is no active edge with $x'_e = 0$, so (5) is void; (c) holds because at every node all incoming active edges give the same value.

Length of the phase. The next edge to become active is e_8 : $\ell_t - \ell_{v_3} - \frac{3}{2}$ equals $-\frac{19}{990}$ at θ_3 and grows at rate $\ell'_t - \ell'_{v_3} = 112 - \frac{584}{13} = \frac{872}{13}$, so it reaches 0 after $\frac{19}{990} \cdot \frac{13}{872} = \frac{247}{863280}$, i.e. at $\theta_4 = \frac{2}{55} + \frac{247}{863280} = \frac{31639}{863280}$. For e_3 the difference $-\frac{4}{275}$ grows at rate $\ell'_{v_2} - \ell'_{v_1} = 21$ and would need $\frac{4}{5775} > \frac{247}{863280}$. No queue can empty during the phase: the differences $\ell_w - \ell_v - \tau_e$ of the edges with a queue are nondecreasing, except for e_0 (rate $\ell'_{v_1} - \ell'_s = -\frac{6}{13}$, difference $\frac{4}{275}$ at θ_3 , so it would need $\frac{26}{825}$ time units to reach 0) and e_6 (rate $\ell'_{v_4} - \ell'_{v_3} = -\frac{304}{13}$, difference $\frac{2}{3}$, so it would need $\frac{13}{456}$), both far longer than the phase.

7 Remarks

Remark 1 (Strictly positive delays). If one prefers all delays to be positive, replace τ_e by $\tau_e + \varepsilon(j - i)$ for every edge $e = (v_i, v_j)$, with any $\varepsilon > 0$ (all edges go from a lower to a higher index). Every path from s to v_j becomes longer by exactly εj , so all labels become $\ell_{v_j}(\theta) + \varepsilon j$, the differences $\ell_w - \ell_v - \tau_e$ and hence all phases and derivatives are unchanged, and both makespans increase by 6ε . The strict inequality $T_{\text{EQ}}^1 < T_{\text{EQ}}^2$ is preserved.

Remark 2 (Simple graph). Replacing the edge $e_3 = (v_1, v_2)$ with $(\nu, \tau) = (4, 2)$ by a path $v_1 \rightarrow v' \rightarrow v_2$ of two edges with $(\nu, \tau) = (4, 1)$ each does not change the equilibrium (Lemma 7 of the paper), and gives a simple graph with eight nodes and twelve edges with the same two makespans.

Remark 3 (Other inflow rates). The same six phases, in the same order, occur for every inflow rate $u_0 > 35485/6824 \approx 5.2$; for such u_0 a symbolic computation gives

$$T_{\text{EQ}}(u_0) = \ell_t\left(\frac{3}{10u_0}\right) = \frac{12383220632 u_0 - 1557689395}{267430410 (8u_0 - 1)}, \quad \frac{dT_{\text{EQ}}}{du_0} = \frac{334592}{1142865 (8u_0 - 1)^2} > 0,$$

so the makespan for $M = 3/10$ is strictly increasing in the inflow rate on the whole range $u_0 > 5.2$, not only between 6 and 7. In every phase the rate at which flow reaches the sink does not depend on u_0 ; what depends on u_0 is when the phases start, and the net effect is that the two improvements of the sink rate caused by e_8 and e_3 reach a smaller or larger part of the flow.

Remark 4 (Where the loss comes from). The slow edge e_1 from s to v_2 becomes active when the waiting time in the queues of e_0 and e_2 reaches 2; with a higher inflow rate this happens for an earlier particle. From then on only part of the flow feeds the queue of e_2 , so the queue of e_2 , which eventually opens the fast edge e_3 , grows more slowly per unit of flow. The result is that with $u_0 = 7$ a larger share of the $M = 3/10$ units has already left s when e_3 opens than with $u_0 = 6$, and this share misses the fastest regime (phase 5). Numerically, for $T_{\text{EQ}}^1 - T_{\text{EQ}}^2$ the earlier activation of e_8 under $u_0 = 7$ contributes about $+7.7 \cdot 10^{-4}$ and the later activation of e_3 about $-8.8 \cdot 10^{-4}$; the net result is $-1.13 \cdot 10^{-4}$.

Remark 5 (Verification). All numbers above were computed in exact rational arithmetic and verified in three independent ways: by checking conditions (a)–(c) phase by phase; by simulating the queues of every edge directly from the edge inflow rates x'_e/ℓ'_v with the queue dynamics of the paper, recomputing the labels $\ell_v(\theta) = \min_e T_e(\ell_u(\theta))$ from these queues, and checking that flow is only sent on active edges; and by a discrete simulation with 10^7 small packets, each choosing a fastest route given the earlier packets, whose makespans converge to the two values above (with the same sign of the difference).