Evaluating priority rules in school choice*

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Abstract

In many school choice programs, students are centrally assigned to schools using the deferred acceptance algorithm. The priority rule employed by schools–such as proximity, sibling attendance, or random scores–significantly impacts the efficiency of the assignment process. We derive upper and lower bounds for the fraction of students assigned to their top schools and the fraction of students that can be Pareto improved in a large market school choice model. Additionally, we characterize the inefficiencies resulting from various priority rules. Our bounds also facilitate the comparison of different priority rules. We apply our bounds to examine random tie breaking rules. We also show that while distance-based priorities may result in several students being assigned to their top schools, they can also lead to significant efficiency losses. Simulations using Chilean data confirm our theoretical findings.

Keywords: market design, school choice, priorities

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1 Introduction

In many school choice programs worldwide, students are centrally assigned to schools using the deferred acceptance (DA) algorithm proposed by Gale and Shapley (1962). This algorithm results in a stable matching that need not be Pareto efficient for students (Roth and Sotomayor, 1990). Moreover, as Kesten (2010) shows, for any fixed supply of seats, it is possible to construct the demand so that the stable matching assigns each student to her worst or second-worst school. This naturally leads to the question of which policy decisions can mitigate the inefficiencies of the DA algorithm.

The priority rule used by schools to rank students is an important implementation decision that impacts the satisfaction that students and families have with the assignment process. In real-world applications, policymakers use a variety of priority criteria (Cantillon et al., 2022). In cities such as Boston and Copenhagen, a student gets priority based on proximity to a school. In New Haven, students get higher priority in schools in which they have siblings. In many cities, schools use random priorities. While in Amsterdam a student gets a random score that applies to all schools, in Chile each student gets a different random score for each school. As Abdulkadiroğlu et al. (2009) and Leshno and Lo (2021) emphasize, the priority structure is critical in school choice implementations and can be even more important than the specific algorithm used to assign students to schools.

This paper provides results to theoretically evaluate the impact of different priority rules on some efficiency measures in school choice programs that employ the DA algorithm. We derive tight upper and lower bounds for the fraction of students assigned to their top schools in a large market school choice model. Our bounds are determined by students' preferences, the supply of seats and the priority rule used by schools to rank students. We thus provide and apply a methodology to evaluate the performance of different priority criteria usually employed in school choice implementations.

We study a large market model in which a continuum of students applies to a finite number of schools (Azevedo and Leshno, 2016; Abdulkadiroğlu et al., 2015). Students have preferences over schools, while schools have priorities over students. To capture the interplay between students' preferences and schools' priorities, each student has a type. These types govern students' preferences and also determine their scores within each school, subsequently influencing the schools' priorities over students. Introducing types into our matching model is a flexible way to allow for correlation between students' preferences and scores. Our setup encompasses a variety of priority criteria used in applied school choice, including multiple tie breaking–in which each student receives a different random number for each school–and single tie breaking–in which a student obtains a unique random score that determines her priorities in all schools. It also accommodates models in which students and schools are geographically differentiated and a student's preference is partly determined by her location.

Our first main result, Theorem 1, provides tight upper and lower bounds for the fraction of students assigned to their top schools in a stable matching. Behind these bounds is the idea that the performance of a stable matching depends on how students can congest and get admission to schools they do not consider top choices. The bounds are easy to apply to examples and models that would otherwise pose significant challenges in analysis.

To establish Theorem 1, we exploit a set of market-clearing conditions that characterize stable matchings using cutoffs (Azevedo and Leshno, 2016; Abdulkadiroğlu et al., 2015). The solutions to these equations are hard to solve in closed form. We thus explore relaxed market-clearing conditions. These relaxed conditions are used to obtain lower and upper bounds for the cutoffs that the original market-clearing conditions. These bounds are then used to estimate the fraction of students assigned to their top schools.

Equipped with Theorem 1, we can characterize the impact of various priority protocols used in school choice. An important literature has studied the role of different random tie breaking rules on the effectiveness of the DA algorithm (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020; Arnosti, 2022; Allman et al., 2022). Proposition 1 shows conditions such that single tie breaking results in more students assigned to their top schools than multiple tie breaking. In contrast to previous results, Proposition 1 applies even when no parametric restriction is imposed on the demand for schools.

We also evaluate distance-based priorities in a general spatial model of school choice. Stable matchings under distance-based priorities are determined by how students value proximity, and also by the capacity and geographical distribution of schools. Under distance-based scores, when students significantly value proximity, students' preferences and schools' priorities are compatible: a student who likes a school also has a high score in the school. As Proposition 2 shows, in this case, the resulting stable matching will be Pareto efficient and place many students into their top schools.

In contrast, distance-based priorities may result in important efficiency losses when students' preferences for proximity are not strong. Indeed, Proposition 3 shows that multiple tie breaking may result in more students assigned to their top schools and fewer students that can be Pareto improved than distance-based priorities. This happens even when students value proximity and, as a result, there is positive correlation between preferences and priorities. We observe that in markets where students care not only about proximity to schools but also about other aspects –such as scores in standardized tests, extracurricular activities, etc– the consistency between preferences and priorities is positive but weak. In these markets, under distance-based priorities, a student may be assigned to a school that is not her top but just happens to live nearby. This force leaves relatively few students assigned to top schools under distance-based priorities.

As intuitive as Proposition 3 may seem, this intuition does not easily translate into a proof, nor is it prominent in the literature. Proposition 3 applies to a market where students are spatially located and have arbitrary preferences over schools. Theorem 1 can be used to provide an otherwise difficult characterization of a stable matching in a general model of spatial differentiation.

Proposition 3 also offers a counterpoint to the conjecture made by Pathak (2017) that proximity reduces the inefficiencies of DA by inducing a positive correlation between preferences and priorities. Proposition 2 confirms this observation when preferences for proximity are strong.[1] In contrast, when preferences are positive but weakly correlated with proximity, Proposition 3 shows that distance-based priorities may lead to significant efficiency losses.

Our theoretical results show that introducing proximity as a priority criterion has ambiguous effects on the fraction of students assigned to their top schools. We confirm our findings by performing simulations using data from the centralized school choice system in Chile.

Related literature. The school choice literature has shown that even the student optimal stable matching need not be Pareto efficient (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Several papers derive conditions under which a stable matching is Pareto efficient.[2] Notably, Ergin (2002) introduces a class of school priorities such that, regardless of students' preferences, the stable matching is efficient.[3] See also Ehlers and Erdil (2010), Salonen and Salonen (2018), Reny (2021), Pakzad-Hurson (2023). In many practical applications of the deferred accepted algorithm, efficiency will not be achieved and therefore understanding the magnitude of inefficiencies may be a useful step when designing priorities in matching markets.

[1] See also Cantillon et al. (2022).

[2] There is also an important set of papers proposing alternative algorithms and solutions, including Shapley and Scarf (1974), Kesten (2010), Che and Tercieux (2018), Ehlers and Morrill (2020), Cantillon et al. (2022), and Reny (2022).

[3] Ergin (2002) introduces acyclical priorities. In school choice applications, priorities derived under single tie breaking are acyclical.

Several papers have observed that school priorities can be designed to impact the performance of the deferred acceptance algorithm. Abdulkadiroğlu et al. (2009), Ashlagi and Nikzad (2020), Arnosti (2022), Shi (2022), Allman et al. (2022) notice that when schools solve indifferences by using random lotteries, the correlation between the scores of a student in different schools is important for efficiency. As the literature shows –and we confirm in Subsection 4.1–single tie breaking (under which the correlation between scores is perfect) results in a more efficient matching than multiple tie breaking (under which the correlation between scores is 0). We make three contributions to this literature. First, Proposition 1 shows that we can compare multiple to single tie breaking even when no assumption is imposed on students' demand. Second, we observe that the correlation between students' preferences and scores is also important to evaluate the efficiency of a stable matching–see Propositions 2 and 3. Third, we notice that priority criteria such that a high score in a school implies low scores in other schools make efficiency hard to achieve; see the discussion of distance-based priorities following Proposition 3. In this sense, priority criteria that result in no correlation between scores (such as multiple tie breaking) may produce a more efficient matching than priority criteria resulting in a negative correlation (such as distance-based priorities).

Our paper also connects to research about distance-based priorities in school choice. Dur et al. (2018) explore how different precedence orders implementing walk-zone reserves impact the fraction of reserve-group students assigned to each school. More closely related, Çelebi and Flynn (2021) show that in a large market model, the optimal coarsening of scores is attained by splitting agents into at most three indifference classes. They also explore a model in which scores are determined by distance and show that the the optimal number of zones depends on the diversity goals of the planner. Our focus is different in that we explore alternative performance measures and our insights highlight how the correlation between preferences and priorities determine the effectiveness of the deferred acceptance algorithm. We thus see our analysis as complementary to Çelebi and Flynn's (2021).

Finally, our work connects to the literature employing large market models to analyze market design questions (Azevedo and Leshno, 2016; Abdulkadiroğlu et al., 2015; Leshno and Lo, 2021; Allman et al., 2022). We provide a method to bound cutoffs for stable matchings in large market models, and derive new insights for priority design in school choice.

Organization of the paper. Section 2 introduces the model. Section 3 presents our bounds for the fraction of students assigned to their top schools. Section 4 applies our bounds to random priorities and distance-based priorities. Section 5 shows simulations using Chilean data. Section 6 presents concluding remarks. All proofs are in the Appendix.

2 Model

2.1 Environment

There is a finite set of schools C with $|C| = N \geq 2$. There is a continuum S of students to be matched to schools. Each student s has a strict preference ordering $\succ^s$ over $C \cup \{\emptyset\}$, where $\emptyset$ is the outcome if s is unassigned. A student s has a score vector $e^s = (e^s_c)_{c=1}^N$. School c has capacity k_c. A school c prefers student s to student s' iff $e^s_c > e^{s'}_c$. We simplify exposition and assume that all schools and all students are acceptable.

Students have types $i \in I$. We endow $I \subseteq \mathbb{R}^L$ with a measure ν so that $\int \nu(di) = 1$ and assume that ν is absolutely continuous. Preferences and scores are determined by types. Concretely, for each i there is a distribution F_i over the finite set of preferences over schools, with $\sum_\succ F_i(\succ) = 1$ and $F_i(\succ) \geq 0$, so $F_i(\succ)$ is the fraction of type i students having preference $\succ$. Additionally, a type i student has a score $e^s_c = e_c(i) \in [0,1]$. We assume that the probability of a tie in a school is 0 so that for all c and all $x \in [0,1]$, $\nu(\{i \in I \mid e_c(i) = x\}) = 0$. Implicit in our model is the assumption that any correlation between the preferences of a student s and her scores in schools is determined by the type i of the student $s \in S$. Since a student's type determines scores, a student s can be characterized by her type i and preferences $\succ$, $s = (i, \succ)$. We denote by $\bar{\nu}$ the measure induced by ν and $(F_i)_{i \in I}$ over the set of students.[4]

Let $F^k_i(c)$ be the fraction of type i students that put school c in the k-th position:

$$F^k_i(c) = \sum_{\succ \text{ such that } c \text{ is ranked } k} F_i(\succ)$$

and $\bar{F}_i(c)$ be the fraction of type i students listing school c: $\bar{F}_i(c) = \sum_{k=1}^N F^k_i(c)$. Since we assume that students list all schools, $\bar{F}_i(c) = 1$ for all i and all c.[5]

[4]Given any subset of students $S' \subseteq S$, $\bar{\nu}(S') = \int \sum_{\succ \text{ such that } (i,\succ) \in S'} F_i(\succ)\nu(di)$.

[5]This assumption simplifies exposition. Our results can be easily accommodated to the case in which $\bar{F}_i(c) < 1$ for some i and c. In the Appendix, we provide results and proofs that apply even when students do not apply to all schools.

We assume that the all schools are popular in the sense that for all c,

$$F^1(c) := \int F_i^1(c)\nu(di) > k_c.$$

Our analysis can be extended to the case in which this inequality holds for some but not all schools, but we simplify exposition by imposing the inequality in all schools. We also assume that $\bar{F}(c) = 1 > F^1(c)$ for all c so that each school has a nontrivial mass of students that demand it but not in the top position.

2.2 Priorities

In this paper, we evaluate different priority criteria. We now discuss how prominent priority rules used in school choice models can be cast as special cases of our general model.

Several papers compare single to multiple tie breaking in school choice problems (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020; Arnosti, 2022; Allman et al., 2022). Our model also accommodates these priorities.

Example 1 (Random tie breakings). *Take a school choice model in which students have no types and students' preferences are given by a distribution $F(\succ)$. Given the set of schools, scores at each school are randomly determined in $[0,1]$. Our general model can accommodate these random priorities as follows.*

Let $I = [0,1]^N$ be the set of types and ν be N independent uniform distributions over $[0,1]$. The c-component of a student type $i \in [0,1]^N$ determines the score that student i has in school c, that is, $e_c(i) = i_c$. In this case, our model becomes a school choice problem in which students are ranked according to multiple tie breaking (MTB) (Abdulkadiroğlu et al., 2009).

The model can also accommodate the case of single tie breaking (STB). When $I = [0,1]$, and ν is the uniform distribution on $[0,1]$. A type i student has score i at each school. Our model becomes a school choice problem in which students are ranked according to a single lottery (Abdulkadiroğlu et al., 2009).

The setup introduced in Subsection 2.1 can also be used to model a school choice environment with geographical differentiation and distance-based priorities (Dur et al., 2018; Çelebi and Flynn, 2021).

Example 2 (Horizontal differentiation and distance-based priorities). *$I \subset \mathbb{R}^2_+$ models a city and a student's type is her location $i \in I$ in the city. Schools are located and spread across the city. Let*

$d(i,c) \in [0,1]$ be a distance between a student located in i and school c.[6] *Similar to Abdulkadiroğlu et al. (2017), the utility that a student located in i derives from attending school c is in part determined by $d(i,c)$. For example, one can generate the utility that a type i student derives from school c as*

$$u_{s,c} = -d(i,c) + \epsilon_{i,c}$$

where $\epsilon_i = (\epsilon_{i,c})_{c=1}^{N}$ is a shock vector and has a distribution H_i.[7] *In this case, we can construct the distribution over the finite set of preferences as:*

$$F_i(\succ) = Prob[u_{s,c1} \geq u_{s,c_2} \geq \cdots \geq u_{s,c_N}]$$

where $c_1 \succ \cdots \succ c_N$.

Schools can rank students using a variety of criteria (including random tie breaking, discussed above). Under distance-based priorities, the score that a student type i has in school c is given by $e_c^s = 1 - d(i,c)$.

2.3 Stable matchings

A matching is a function $\mu : S \cup C \to (C \cup \{\emptyset\}) \cup 2^S$ such that:

i. For all $s \in S$, $\mu(s) \in C \cup \{\emptyset\}$;

ii. For all $c \in C$, $\mu(c) \subseteq S$ with $\bar{\nu}\big(\{s|\mu(s) = c\}\big) \leq k_c$;

iii. For all $c \in C$ and all $s \in S$, $\mu(s) = c$ iff $s \in \mu(c)$.

iv. For all c, $\{s \in S \mid c \succ_s \mu(s)\}$ is open.

The first condition says that each student is assigned to a school. The second condition says that each school is assigned to a measure of students that does not exceed its capacity. The third condition says that a student is assigned to a school iff the school is assigned to that student. The fourth condition

[6] The distance function can be arbitrary. The only property relevant for our analysis is that the distance function $d(i,c)$ satisfies the triangle inequality.

[7] Using this formulation, we can model fixed effects and also interaction effects other than distance (Abdulkadiroğlu et al., 2017).

is technical and eliminates redundant matchings that differ in a measure 0 of students (Azevedo and Leshno, 2016).

A matching μ is stable if for all $s \in S$ and all $c \in C$ such that $c \succ_s \mu(s)$, the following conditions hold: (i) $\bar{\nu}\big(\{s|\mu(s) = c\}\big) = k_c$; and (ii) $e_c^s < e_c^{s'}$ for all s' with $\mu(s') = c$. Intuitively, a matching is stable if there is no pair (s, c) that can block the matching (Gale and Shapley, 1962). Stability is an important desideratum in matching theory and its many applications (Roth, 1982; Abdulkadiroğlu et al., 2009).

To characterize stability, we follow Abdulkadiroğlu et al. (2015) and Azevedo and Leshno (2016) and find stable matchings as solutions to a supply and demand system of equations. Given cutoffs $p = (p_c)_{c=1}^N$, a student s can get admission to c if $e_c(i) \geq p_c$. A student demands her favorite school among those she can get admission given p. We thus define $D_c(p)$ as the measure of students that demand school c as a function of cutoffs p.[8] A stable matching can be found by means of market-clearing cutoffs $p = (p_c)_{c=1}^N$ that solve

$$D_c(p) = k_c \quad \forall c \tag{2.1}$$

Given market-clearing cutoffs, a stable matching is built by assigning each student to her most preferred school among those where her score exceeds the cutoff.

While the system of equations (2.1) is neat and simple to interpret, it can be solved in closed-form solutions only for special cases. When we can find a closed-form solution to (2.1), it is simple to calculate statistics for the resulting stable matching. However, solving the model in closed-form is unfeasible even for relative simple models.[9]

3 Students assigned to their top schools

This Section states and discusses our bounds for the measure of students assigned to their top schools. We then provide some examples and sketch some of the arguments in the proof.

For a given matching, let $R^1(c)$ be the mass of students assigned to school c that put c as their top school. Obviously, $0 \leq R^1(c) \leq k_c$. $R^1(c)$ is an important metric usually employed by policy makers to

[8] We provide a formula for $D_c(p)$ in the Appendix; see Equation (A.1).

[9] The system of equations (2.1) is non-linear in p. Under multiple tie breaking, each equation in (2.1) is polynomial of degree N.

evaluate the effectiveness of a matching (Abdulkadiroğlu et al., 2009). In the next Section, we discuss other performance measures.

For each school c, we compute the demands

$$\Lambda_c^1(x) = \int_{e_c(i)\geq x} F_i^1(c)\nu(di) \quad \text{and} \quad \bar{\Lambda}_c(x) = \int_{e_c(i)\geq x} \nu(di)$$

for all $x \in [0,1]$. Let $\phi_c \in [0,1]$ and $\Phi_c \in [0,1]$ be defined by the equations

$$\phi_c = \max\big\{x \in [0,1] \mid \Lambda_c^1(x) = k_c\big\} \tag{3.1}$$

$$\Phi_c = \min\big\{x \in [0,1] \mid \bar{\Lambda}_c(x) = k_c\big\}. \tag{3.2}$$

In contrast to the cutoff p_c that clears the market for school c in a stable matching, cutoffs ϕ_c and Φ_c are entirely determined by the local demand for school c: while ϕ_c is determined by the mass of students that demand c first $(F_i^1(c))_i$, Φ_c is determined by the mass of students that list c in any position $(\bar{F}_i(c))_i$.

The following result provides estimates for $R^1(c)$ in a stable matching.

Theorem 1. *For any stable matching and all $c = 1, \ldots, N$:*

$$R^1(c) \geq k_c - \bar{\eta}_c\Big(\int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ for some } \hat{c}\neq c} (1 - F_i^1(c))\nu(di)\Big) \tag{3.3}$$

and

$$R^1(c) \leq \int_{e_c(i)\geq\Phi_c} F_i^1(c)\nu(di) + \eta_c\Big(\int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)\geq\phi_{\hat{c}} \text{ for some } \hat{c}\neq c} (1 - F_i^1(c))\nu(di)\Big) \tag{3.4}$$

where

$$\bar{\eta}_c = \min\Big\{1, \frac{\inf_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\Lambda_c^1(x)}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\big(\Lambda_c^1(x) + \int_{e_c(i)\geq x, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\neq c}(1 - F_i^1(c))\nu(di)\big)}\Big\}$$

and

$$\eta_c = \frac{\inf_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\Lambda_c^1(x)}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\bar{\Lambda}_c(x)}.$$

Both bounds are tight.

Theorem 1 provides estimates for the measure of students assigned to their top schools. To apply the bounds, we compute cutoffs ϕ_c and Φ_c that are entirely determined by each school c supply and demand. The real numbers η_c and $\bar{\eta}_c$ adjust for the fact that we do not employ stable matching cutoffs p but approximated cutoffs ϕ_c and Φ_c. As we show in Section 4, getting simple expressions for η_c, $\bar{\eta}_c$, ϕ_c and Φ_c for specific models is straightforward. Theorem 1 can be easily applied to obtain substantive insights for several examples and models.

The idea behind bounds (3.3) and (3.4) is that the measure of students assigned to their top schools depends on how students can congest schools they do not rank top. A student that does not rank a school c at the top may still congest it depending on her score in c and, critically, her scores in schools $\hat{c} \neq c$. Thus, the measure of students assigned to their top schools critically depends on how types determine preferences for each school and scores across schools. We now discuss each of the bounds.

A. Discussion of lower bound. The first bound in the Theorem, inequality (3.3), provides a condition under which a high fraction of students assigned to school c will rank it as their top school. Most students will be assigned to their top school in c when (i) students that rank c apply to c first, that is, $1 \approx F_i^1(c)$; or (ii) students that rank c second, third, etc have a low score in c, that is, $\int_{e_c(i)\geq\phi_c}(1 - F_i^1(c))\nu(di) \approx 0)$; or, more generally, (iii) most students that rank c second, third, etc and have a high score in c are also likely to get admission in some other school, that is, $\int_{e_c(i)\geq\phi_c}(1 - F_i^1(c))\nu(di) \approx \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)\geq\Phi_{\hat{c}} \text{ all } \hat{c}\neq c}(1 - F_i^1(c))\nu(di)$.[10] In general, to evaluate inequality (3.3), we compute the measure of the set of students that rank c second, third, etc, and have a high score in c and a low score in some other school $\hat{c}$. When this measure is low, most students that get admission to c will naturally rank c top.

Inequality (3.3) can be used to derive conditions under which all students assigned to a school rank the school top.

Corollary 1. *Suppose that for all i and c such that $F_i^1(c) < 1$ and $e_c(i) \geq \phi_c$, we have that $e_{\hat{c}}(i) \geq \Phi_{\hat{c}}$ for all $\hat{c} \neq c$. Then, $R^1(c) = k_c$ for all c and a stable matching is Pareto efficient.*

The following example shows a stable matching that results in all students assigned to their top schools. This happens even when preferences and priorities do not conform: in our example, some

[10] Note that

$$\int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\neq c}(1 - F_i^1(c))\nu(di) = \int_{e_c(i)\geq\phi_c}(1 - F_i^1(c))\nu(di) - \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)\geq\Phi_{\hat{c}} \text{ all } \hat{c}\neq c}(1 - F_i^1(c))\nu(di)).$$

students rank a school top, but that school does not rank those students highly.[11] The example below also shows that the lower bound (3.3) is tight.

Example 3. *Suppose that* $N = 2$ *and* $I = [0, 1]$*. Each school has capacity* $k = \frac{1}{4}$*. Students* $i \leq 1/4$ *are elite students, with outstanding academic performance. For* $i \leq 1/4$*, scores are given by* $e_{c_1}(i) = e_{c_2}(i) = 1 - i$*. School* c_1 *(resp.* c_2*) is located in 0 (resp. 1) and students* $i > 1/4$ *are ranked according to distance. Concretely, for* $i > 1/4$ $e_{c_1}(i) = 1 - i$ *while* $e_{c_2}(i) = i - 1/4$*. For each* i*, a fraction* $\alpha(i)$ *(resp.* $1 - \alpha(i)$*) of students rank school* c_1 *first (resp. school* c_2 *first). Assume that* $\alpha(i) = 1$ *for* $i \leq 1/2$ *and* $\alpha(i) = 0$ *for* $i > 1/2$*.*

It is simple to see that $\phi_{c_1} = 3/4$*,* $\Phi_{c_1} = 3/4$*,* $\phi_{c_2} = 1/2$*,* $\Phi_{c_2} = 3/4$*. Note that all students that rank school* c_1 *second have scores* $e_{c_1} < 1/2 < 3/4$*. All students that rank* c_2 *second and have scores* $e_{c_2}(i) > \phi_{c_2} = 1/2$ *also have score* $e_{c_1}(i) > \Phi_{c_1} = 3/4$*. Using Corollary 1,* $R^1(c_1) = R^1(c_2) = k$*.*

B. Discussion of upper bound. The second bound in Theorem 1, inequality (3.4), provides a condition under which a low fraction of students assigned to school c will rank it as their top school. The Theorem shows that in a stable matching, few students assigned to school c will rank it as their top school when (i) most students that rank c top are unlikely to have sufficiently high scores, that is, $\sup\{F_i^1(c) \mid e_c(i) \geq \Phi_c\} \approx 0$, and (ii) most students that rank c second, third, etc and have a high score in c are unlikely to get admission in some other school, that is, $\int_{e_c(i)\geq\phi_c}(1 - F_i^1(c))\nu(di) \approx \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)\leq\phi_{\hat{c}} \text{ all } \hat{c}\neq c}(1 - F_i^1(c))\nu(di)$. If (i) were not satisfied, then we could find a non-negligible mass of students for whom c is the top choice and are sure to be assigned. Condition (ii) ensures that students for whom c is listed but is not top are admitted to c in a stable matching.

We now illustrate the upper bound. The next example shows that the upper bound (3.4) is tight.

Example 4. *Suppose that* $N = 2$ *and* $I = [0, 1]$*. Each school has capacity* $k = \frac{1}{4}$*. Students* i *live in position* i *with preferences given by* $F_i^1(c) = 1/2$ *and* $\bar{F}_i^1(c) = 1$ *for each* c*. Schools* c_1 *and* c_2 *are located at the extremes of the interval. Priorities are distance-based so the scores of agent* i *are given by* $e_{c_1}(i) = 1 - i$ *and* $e_{c_2}(i) = i$*. It is simple to see that* $\phi_c = 1/2$ *and* $\Phi_c = 3/4$*. Thus, inequality (3.4)*

[11] Erdil and Ergin (2008) show simulations in which the preferences of both sides of the market conform and, as a result, the stable matching is efficient. In those simulations, priorities are given by multiple tie breaking and walk zones. As distance becomes more important for students (in their model, that is captured by $\beta \to 1$), the efficiency loss in the stable matching goes to 0 since in the limit both sides of the market have perfectly conforming preferences. See also Salonen and Salonen (2018) for theoretical results. See also Proposition 2.

becomes

$$R^1(c) \leq \int_{1-i\geq 1-k} \frac{1}{2}di + \frac{1}{2}\int_{1-i\geq 1/2, i\geq 1/2} \frac{1}{2}di = \frac{k}{2}.$$

In the unique stable matching, the cutoff equals $p_c = 3/4$*, and thus in each school only half of the students assigned to the school rank the school first:* $R^1(c) = k/2$*. It thus follows that inequality (3.4) holds with equality.*

3.1 Proof sketch for Theorem 1

We close this Section by discussing the main ideas behind the proof of Theorem 1. Since ϕ_c solves a market-clearing condition for a demand Λ_c^1 that is below the total demand D_c, we deduce that $\phi_c \leq p_c$ for any cutoff vector p from a stable matching. Analogously, $p_c \leq \Phi_c$. See Lemma 1 in the Appendix for details.

Cutoffs ϕ_c and Φ_c are important in that they provide bounds for cutoffs p characterizing stable matchings. More subtly, ϕ_c and Φ_c are informative about the measure of students assigned to their top schools. Indeed, when $p_c = \phi_c$, then the number of students assigned to their top schools in c equals k_c:[12]

$$R^1(c) = \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) = \int_{e_c(i)\geq \phi_c} F_i^1(c) = k_c.$$

Similarly, we note that

$$R^1(c) = \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) = \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) + \int_{p_c\leq e_c(i)\leq \Phi_c} F_i^1(c)\nu(di).$$

It follows that $R^1(c) = 0$ when $\Phi_c = p_c$ and $F_i^1(c) = 0$ for all $e_c(i) \geq \Phi_c$.

In general, however, $\phi_c < p_c < \Phi_c$. The key technical observation that enables us to prove Theorem 1 is that we can bound $p_c - \phi_c$ and $\Phi_c - p_c$. Indeed, we bound $p_c - \phi_c$ and $\Phi_c - p_c$ by using several relaxed market-clearing conditions. More technically, the proof bounds the distance between the solutions to different non-linear market-clearing equations to derive estimates for the measure of students assigned to their top schools. See the Appendix for details.

[12] Moreover, $R^1(c) = k_c$ iff $\phi_c = p_c$.

4 Priorities in school choice

We now explore the impact of different priority structures on the fraction of students assigned to their top schools and efficiency. The setup for this Section is the model of horizontal differentiation presented in Example 2. We fix the demand and the capacity of each school and compute the bounds from Theorem 1 for different priorities. Subsection 4.1 explores random multiple and single tie breaking priorities. Subsection 4.2 explores distance-based priorities. Subsection 4.3 compares multiple tie breaking to distance-based priorities. We denote the fraction of students assigned to their top schools under distance-based priorities, multiple tie breaking, and single tie breaking by R^1_{DB}, R^1_{MTB} and R^1_{STB}, respectively.

4.1 Random priorities

This Subsection applies our bounds to the widely studied model of school choice with random priorities (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020; Arnosti, 2022; Allman et al., 2022).

It is simple to see that under single or multiple tie breaking, cutoffs are identical and given by

$$\phi_c^{RP} = 1 - \frac{k_c}{F^1(c)} \quad \text{and} \quad \Phi_c^{RP} = 1 - k_c \tag{4.1}$$

For multiple tie breaking, we can also compute

$$\bar{\eta}_c = \frac{F^1(c)}{F^1(c) + (1 - F^1(c))\left[1 - \prod_{\hat{c}\neq c} k_{\hat{c}}\right]} \quad \text{and} \quad \eta_c = F^1(c). \tag{4.2}$$

Using Theorem 1, in Appendix B.1.1 we deduce that under multiple tie breaking, for each school c[13]

$$\begin{aligned} & k_c\left(\frac{F^1(c)}{F^1(c) + (1 - F^1(c))\big(1 - \prod_{\hat{c}\neq c} k_{\hat{c}}\big)}\right) \\ \leq \quad & R^1_{MTB}(c) \leq \quad k_c\left(1 - (1 - F^1(c))\prod_{\hat{c}\neq c}\left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right). \end{aligned} \tag{4.3}$$

[13]For multiple tie breaking, is is possible to derive bounds that do not use Theorem 1. By definition,

$$R^1_{MTB}(c) = F^1(c)(1 - p_c) \in [k_c(1 - \Phi_c), k_c(1 - \phi_c)] = [k_c F^1(c), k_c].$$

The bounds given in (4.3) are strictly sharper than these simple bounds.

Since $\bar{F}(c) = 1 > F^1(c)$, both the upper and the lower bound for $R^1_{MTB}(c)$ are informative. Under multiple tie breaking, some students will be assigned to school c even when c is not their top school, but there will always be some students assigned to their top schools.

We can also apply Theorem 1 to a model with single tie breaking (see Appendix B.1.2). We can then compare multiple and single tie breaking.

Proposition 1. *Suppose that*

$$\frac{k_c}{F^1(c)} \leq \frac{\min\{k_{\hat{c}} \mid \hat{c} \neq c\}}{1 - F^1(c)\prod_{\hat{c}\neq c}(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})})}. \tag{4.4}$$

Then,

$$R^1_{STB}(c) \geq R^1_{MTB}(c) \tag{4.5}$$

This bound says that when school c is sufficiently popular (that is, $\frac{k_c}{F^1(c)}$ is small enough), more students are assigned to c in the top position under single tie breaking than under multiple tie breaking. Proposition 1 is similar to the results obtained by Ashlagi and Nikzad (2020), Arnosti (2022), and Allman et al. (2022). These studies compare the number of students assigned to their top schools under multiple and single tie-breaking mechanisms. However, Proposition 1 differs in two key respects: (i) it imposes no specific functional form or restrictions on the demand for schools (the papers above assume uniform or multinomial logit models), and (ii) it incorporates a constraint on the capacity of school c.[14]

4.2 Distance-based priorities

Under distance-based priorities, school c ranks students according to $e^s_c = 1 - d(i, c)$. In this Subsection, we argue that the fraction of students assigned to their top schools depends on several factors, including how much students value proximity, and the capacity and geographical dispersion of schools.

To derive a lower bound for $R^1_{DB}(c)$, it is useful to consider the set of all students that can get admission to c given cutoff ϕ^{DB}_c but are rejected by some school $\hat{c}$ given $\Phi^{DB}_{\hat{c}}$:

$$H(c) = \Big\{i \mid d(i, c) \leq 1 - \phi^{DB}_c \text{ and } d(i, \hat{c}) > 1 - \Phi^{DB}_{\hat{c}} \text{ some } \hat{c} \neq c\Big\}.$$

[14]To be clear, most related results impose popularity. We are imposing popularity and that the capacity of schools is small enough.

$H(c)$ estimates the set of all the students that could get admission to c but would be rejected by some school $\hat{c}$. In Appendix B.1.3, we employ Theorem 1 to deduce:

$$R^1_{DB}(c) \geq k_c - \nu\big(H(c)\big) \sup_{d(i,c)\leq 1-\phi_c^{DB}} (1 - F_i^1(c)) \tag{4.6}$$

This bound shows two forces that make $R^1_{DB}(c)$ close to k_c.

A. Consistent preferences and priorities. The first force that makes $R^1_{DB}(c)$ close to k_c follows from a well known observation. When all students living within distance $1 - \phi_c^{DB}$ of school c list c at the top,[15] then $R^1_{DB}(c) = k_c$. In this case, preferences and priorities are consistent in the sense that students that have a high score in c (in other words, that live close to c) also rank school c at the top. When preferences and priorities are consistent, all students will be assigned to their top school and the matching will be Pareto efficient. The observation that consistent preferences and priorities favor efficiency is not new and is discussed by Salonen and Salonen (2018), Echenique et al. (2020), Leshno and Lo (2021), and Cantillon et al. (2022).

B. Clustered schools. The second force that makes $R^1_{DB}(c)$ close to k_c follows by noting that when $\nu(H(c))$ is small, then $R^1_{DB}(c)$ is close to k_c. Intuitively, when $\nu(H(c))$ is close to 0, most students with a score high enough for school c also score high enough in other schools $\hat{c}$. In this case, schools are clustered and distance-based priorities result in a subset of students who are close to all schools and can get admission anywhere. As a result, many of those students get accepted to the school they like the most.[16] The following example illustrates this idea.

Example 5 (Clustered schools)**.** *Suppose that $N = 2$ and $I = [0, 1]$. Each school has capacity $k < 1/2$. A fraction $\alpha(i)$ (resp. $1 - \alpha(i)$) of students rank school c_1 first (resp. school c_2 first) and we assume that $\alpha(i) = 1 - \alpha(1 - i)$ for all $i < 1/2$. Both schools are located at $1/2$. In the unique stable matching, $p_{c_1} = p_{c_2} = k$. Students in $\tilde{I} = [\frac{1}{2} - p, \frac{1}{2} + p]$ could get accepted to both schools and thus*

$$R^1_{DB}(c_1) = R^1_{DB}(c_2) = k.$$

[15] That is, when for all i such that $d(i, c) \leq 1 - \phi_c^{DB}$, $1 = F_i^1(c)$.

[16] This intuition is similar to the idea that under single-tie breaking, many students are assigned to their top schools (Allman et al., 2022).

C. Strong competition and weak preferences for location. We now use Thereom 1 to derive an upper bound for $R^1_{DB}(c)$. Consider the set of all students that could get admission to c and $\hat{c}$ given cutoffs ϕ_c^{DB} and $\phi_{\hat{c}}^{DB}$:

$$A(c,\hat{c}) = \left\{ i \mid d(i,c) \leq 1 - \phi_c^{DB} \right\} \cap \left\{ i \mid d(i,\hat{c}) \leq 1 - \phi_{\hat{c}}^{DB} \right\}. \tag{4.7}$$

Note that if $d(i,c) \leq 1 - \phi_c^{DB}$, by the triangle inequality, $d(i,\hat{c}) \geq d(c,\hat{c}) - d(i,c) \geq d(c,\hat{c}) - 1 + \phi_c$. So, the set $A(c,\hat{c})$ in equation (4.7) is empty whenever

$$2 \leq d(c,\hat{c}) + \phi_c^{DB} + \phi_{\hat{c}}^{DB}. \tag{4.8}$$

The triangle inequality used to derive this condition captures an important intuition about congestion under distance-based priorities: When cutoffs in schools are high and schools are not clustered, having a score high enough for some school implies that the scores in other schools are below the cutoffs. This means that under distance-based priorities, students located near a school will have limited chances to attend other schools. As we show below, this force makes efficiency under distance-based priorities harder to achieve.

Condition (4.8) holds for all schools provided that for all c

1. The function $x \in [0,1] \mapsto \int_{d(i,c)<x} F_i^1(c)\nu(di)$ has strictly positive derivative at $x = 0$;
2. $d(c,\hat{c}) > 0$ for all $\hat{c} \neq c$; and
3. k_c is small enough.

The first condition says that each school has some demand that is arbitrarily close to it. It is relatively simple to show that under the first condition, $\phi_c^{DB} \to 1$ as $k_c \to 0$. Since $d(c,\hat{c}) > 0$, it follows that (4.8) holds when all capacities $(k_c)_{c=1}^N$ are small enough.

Under (4.8), it is simple to apply Theorem 1 to obtain:

$$R^1_{DB}(c) \leq k_c \sup_{d(i,c) \leq 1 - \Phi_c^{DB}} F_i^1(c). \tag{4.9}$$

When capacities are low, under distance-based priorities some students get assigned to a nearby school that is not their top choice. This puts an upper bound on the fraction of students assigned to their most preferred schools.[17]

The next example shows that under (4.8), it is entirely possible that an arbitrarily small fraction of students are assigned to their top schools.

Example 6. *Suppose that $N = 2$ and $I = [0, 1]$. Each school has capacity $k < 1/2$. A fraction $\alpha(i)$ (resp. $1-\alpha(i)$) of students rank school c_1 first (resp. school c_2 first) and we assume that $\alpha(i) = 1-\alpha(1-i)$ for all $i < 1/2$. Assume that $\alpha(i)$ is increasing in i with $\alpha(i) > 0$ for all $i \in [0, 1]$. This means that students tend to value schools that are farther away. Schools rank using distance-based priorities. Under*

$$\int_{i\leq 1/2} \alpha(i) > k. \tag{4.10}$$

it follows that $\phi_{DB}(c) > 1/2$ and (4.8) holds. Since $\bar{\Lambda}(x) = 1 - x$, it is simple to see that $\Phi = 1 - k$. Clearly,

$$R^1_{DB}(c) \leq k \sup_{i\leq k} \alpha(i) = k\alpha(k)$$

It follows that for any $\epsilon > 0$, there exists an increasing function α and $k < 1/2$ such that (4.10) holds and $R^1_{DB}(x) < \epsilon$ for all c.[18]

4.3 Comparing priorities

We now compare distance-based priorities and multiple tie breaking. We evaluate these priority criteria using the fraction of students assigned to their top schools and the fraction of students that can be Pareto-improved. Given any matching μ, a positive measure set of students $S' \subseteq S$ can be Pareto-improved if there exists a matching $\bar{\mu}$ such that for almost all $s \in S$, $\bar{\mu}(s) \succeq_s \mu(s)$ with strict preferences for $s \in S'$. When the matching $\bar{\mu}$ is such that $\bar{\mu}(c) = \mu(c)$ for all $c \in C \setminus \{c', c''\}$, with $c' \neq c''$, we say that S' is part of Pareto-improving pairs. Define

$$P = \bar{\nu}\Big(\bigcup_{S' \text{ can be Pareto-improved}} S'\Big) \quad \text{and} \quad P^2 = \bar{\nu}\Big(\bigcup_{S' \text{ is part of Pareto-improving pairs}} S'\Big)$$

[17] Clearly, the bound is non-trivial only when some of the students living close to c list the school not in the top.

[18] Take $\alpha(i) \geq \epsilon i$ for all $i \in [0, 1/2]$ and $k < \epsilon/2$. Then, $R^1 \leq k\alpha(k) < \epsilon/2 < \epsilon$.

When $P = 0$, the measure of students that can be Pareto-improved is 0 and thus the matching is Pareto-efficient. More generally, P provides the measure of all students who could envision a Pareto-improvement of the proposed matching μ and thus P is a metric of the efficiency of the matching.[19]

Proposition 2. *Suppose that for all c and all i such that $d(i,c) \leq 1 - \phi_c^{DB}$, $F_i^1(c) = 1$. Then, for all c*

$$R_{DB}^1(c) = k_c \quad \text{and} \quad \text{and } P_{DB}^2 = P_{DB} = 0.$$

In particular, no alternative priority criterion can result in more students assigned to top schools than distance-based priorities.

This result shows that when students value distance strongly, then all students are assigned to their top schools under distance-based priorities. To see how the sufficient conditions can be satisfied, fix I, ν, the set of schools C, the distance function $d(i,c)$, and the capacities k_c. For $c \in C$, compute Φ_c^{DB} and assume that capacities are low enough so that $\{i \mid e_c(i) \geq \Phi_c\} \cap \{i \mid e_{\hat{c}}(i) \geq \Phi_{\hat{c}}\} = \emptyset$ for all $c \neq \hat{c}$. Now, construct F_i such that for all i with $e_{\hat{c}}(i) \geq \Phi_{\hat{c}}$, $F_i^1(c) = 1$. This implies that $F_i^1(c) = 1$ for all $i \in I_c$ and therefore $\phi_c^{DB} = \Phi_c^{DB}$ and

$$\sup_{d(i,c) \leq 1 - \phi_c^{DB}} 1 - F_i^1(c) = 0.$$

The following result shows that multiple tie breaking may result in more students assigned to their top schools than distance-based priorities.

Proposition 3. *Assume condition (4.8) and that for all c,*

$$\sup_{d(i,c) \leq 1 - \Phi_c^{DB}} \Big(F_i^1(c) - F^1(c) \Big) \leq \frac{\big(1 - F^1(c)\big)\big(\prod_{\hat{c} \neq c} k_{\hat{c}}\big)}{1 + (\frac{1}{F^1(c)} - 1)\big(1 - \prod_{\hat{c} \neq c} k_{\hat{c}}\big)} \tag{4.11}$$

Then, for all c

$$R_{DB}^1(c) < R_{MTB}^1(c).$$

If we additionally assume that $\mathbb{P}[c \succ c' \mid i] > 0$ for all $c \neq c'$ and all $i \in I$, then

$$P_{MTB}^2 = P_{MTB} < P_{DB} = P_{DB}^2.$$

[19]If S' and S'' can be Pareto-improved, it does not follow that $S' \cup S''$ can be Pareto-improved.

This result provides conditions under which multiple tie breaking assigns more students to their top schools than distance-based priorities. Note that when types do not determine preferences, that is $F_i(\succ) = F(\succ)$ for all $i \in I$, then the left-hand side of (4.11) equals 0 and thus condition (4.11) holds. More generally, condition (4.11) captures the idea that location has a limited impact on preferences so that $F_i^1(c)$ stays relatively flat as a function of i and close to its average $F^1(c)$.[20] Behind this result is the idea that when preferences for nearby schools are weak and competition is strong, distance-based priorities assign some students to schools just because they live nearby even when those schools are not ranked top by them, while under multiple tie breaking those students still have a chance to get accepted in their top schools.

Our results also compare the fractions of students that can be Pareto improved. It is relatively simple to prove that for any matching μ, $P^2 \leq P$ and

$$\sum_{c=1}^{N} R^1(c) + P \leq \sum_{c=1}^{N} k_c. \tag{4.12}$$

We then prove that, under some conditions, these inequalities bind and therefore the fraction of students assigned to top schools and the fraction of students that can be Pareto-improved add up to the total capacity of schools. See Appendix B.2.[21]

5 Simulations

We now validate our theoretical results using data from the centralized public school allocation system in Chile. Our aim is to showcase the main theoretical results from Section 4 by comparing the deferred acceptance outcomes under different priority schemes.

We focus on the admission process for Pre-Kinder in Santiago RM (the main urban center in Chile) for

[20] Note that both conditions (4.11) and (4.8) restrict F and k. They simultaneously hold when types have a limited impact on preferences so that for all c

$$\sup_i F_i^1(c) \leq \frac{F^1(c)}{F^1(c) + (1 - F^1(c))\left(1 - \prod_{\hat{c} \neq c} \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)}$$

and, given preferences, k_c is small enough so that (4.8).

[21] Inequality (4.12) can be strict in some important cases. For example, under single tie breaking the matching is Pareto-efficient, but it is possible that not all students are assigned to their top schools. Example 7 in Appendix B.2 shows that under distance-based priorities, inequality (4.12) can be strict when condition (4.8) does not hold.

2021.[22] Our data contains students' applications (preferences), schools' priorities (scores) and schools' capacities. We also observe students' and schools' geographical location. Our dataset contains $40,921$ students and $1,481$ schools. Since some of our main results (Propositions 1 and 3) apply to tight markets in which the number of seats is not large, in all our simulations, we reduce all schools' capacities by 90%.

We simulate the market under different priority structures. Concretely, we run the deferred acceptance algorithm under multiple and single tie breaking (MTB and STB), and distance-based (DB) priorities. Table 1 shows the fraction of students assigned to their top schools for each priority rule.

	MTB	STB	DB
Fraction of students assigned to top schools	0.044	0.081	0.070
	(0.043, 0.046)	(0.079, 0.082)	-

Table 1: Fraction of students assigned to their top schools. For multiple and single tie breaking, we run 100 simulations. In brackets, we report the 5th and 95th percentiles of the simulations.

As Proposition 1 and the literature show (Abdulkadiroğlu et al., 2009), Table 1 confirms that multiple tie breaking is outperformed by single tie breaking. In turn, distance-based priorities outperform multiple tie breaking. Intuitively, proximity is important when students rank schools.[23]. As a result, under distance-based priorities, preferences and priorities relatively conform. However, single tie breaking still outperforms distance-based priorities.

Proposition 3 applies to markets in which proximity is not a decisive factor when applying to schools. In our dataset, however, students do value proximity. To address this, we modify our dataset to build a market in which students' demand is not explained by proximity to schools. Specifically, we simulate an economy where we maintain students' applications and schools' seats but relocate each school by randomly and independently choosing a point at a given radius $r \geq 0$ from its original location.[24] When $r = 0$, our simulated economy coincides with our original dataset. For $r > 0$, we keep all the rank order lists unchanged, but by randomly relocating schools, the schools in a student's list need not be close to their home.

Figure 1 shows the simulations. Notably, when r is close to 5 km, more students are assigned to

[22] Data is publicly available at http://datos.mineduc.cl/dashboards/20514/descarga-bases-de-datos-sistema-de-admision-escolar/.

[23] There is significant evidence showing that school proximity is a decisive factor in students' preferences (Abdulkadiroğlu et al., 2017) In particular, Aramayo (2018) confirms these findings for the Chilean system.

[24] That is, we fix $r \geq 0$ and each school is relocated by randomly choosing an angle.

their top schools under multiple tie breaking than under distance-based priorities. This holds true even when proximity to schools matters –the city has a diameter of 200 km, and in the simulation, students apply to schools that are on average within 2.6 km from their homes. Thus, we confirm the insights from Proposition 3.

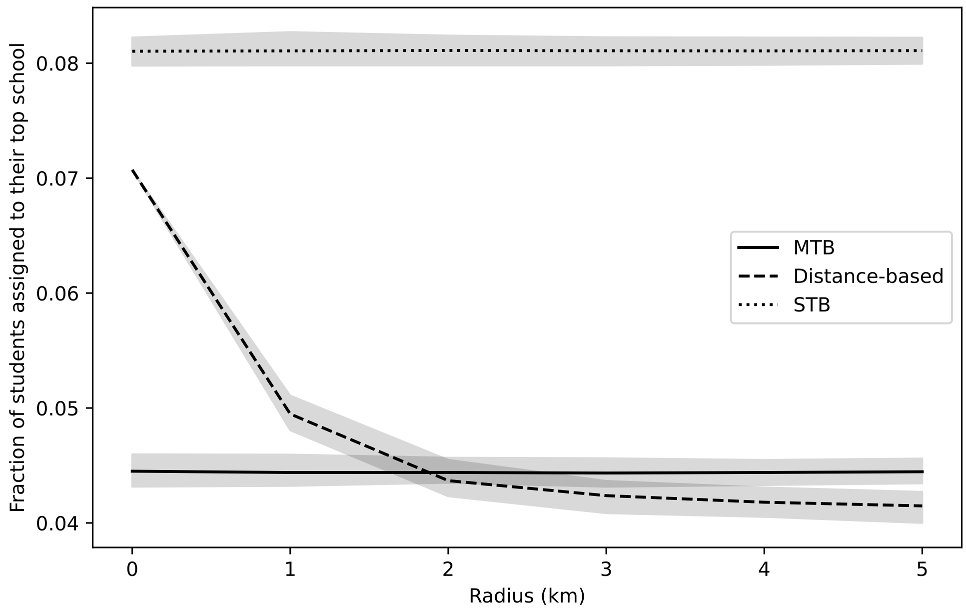

Figure 1: Simulation results. For each $r > 0$, we simulate 100 markets where all schools are randomly relocated using this radius. For each one of these markets, we run the DA algorithm for MTB, STB, and DB priorities.[25] The figure shows the average fraction of students assigned to their top school along different radii. The shaded regions correspond to the simulation fluctuations constructed using the 5th and 95th percentiles.

6 Concluding remarks

This paper provides tight lower and upper bounds for the measure of students assigned to top schools. These bounds apply to a general large market matching model and can be easily applied to different environments. We use our results to examine the impact of various priority rules on stable matchings. In particular, while several school districts employ proximity as a priority criterion, little is known about how this policy decision affects the outputs of the deferred acceptance algorithm. We show that when students highly value proximity, efficient outcomes are achievable. However, under weaker proximity

[25]For MTB and STB, we generate a single random priority draw for each simulated market.

preferences, even multiple tie breaking may assign more students to their top choices than distance-based priorities.

Stable matchings are hard to analyze because comparative statics results and closed-form formulas are typically unfeasible. Future research could sharpen our bounds. Estimating other performance measures, including different diversity metrics, would also be interesting. We leave these research questions for future work.

Appendix

In this Appendix, we provide proofs and supporting material for the main results in the text. In the main text, we simplified exposition and assumed that all students ranked all schools. We relax this assumption. We introduce some notation. Denote the set of schools listed by type i students by $\text{supp}(i) = \{c \mid \bar{F}_i(c) > 0\}$. We abuse notation and for $c \in C$ we denote

$$\text{supp}(c) = \Big\{\hat{c} \in C \setminus \{c\} \mid \exists i \in I \colon c \in \text{supp}(i), \hat{c} \in \text{supp}(i)\Big\}$$

the set of schools that are listed by types that also list c. Note that $\text{supp}(i)$ may not coincide with C. If that is the case, there is some c such that $\bar{F}(c) < 1$.

A Proof of Theorem 1

Define $P_{k,c}$ as the set of all orderings $\succ$ such that school c is listed in position k. Given cutoffs $p \in [0,1]^N$, the demand for school c can be written as

$$D_c(p) = \int_{e_c(i) \geq p_c} F_i^1(c)\nu(di) + \sum_{k=2}^{N} \sum_{\succ \in P_{k,c}} \int_{e_c(i) \geq p_c, e_{\hat{c}}(i) < p_{\hat{c}} \forall \hat{c} \succ c} F_i(\succ)\nu(di) \tag{A.1}$$

The demand is built as follows. Fix a student type i that has school c as its k-th preference. For each one of these student types, a mass $F_i(\succ)$ reveals preference ordering $\succ \in P_{k,c}$. However, only a fraction of $F_i(\succ)$ effectively demands school c. These are students rejected at all $k-1$ schools preferred over c according to $\succ$ ($e_{\hat{c}}(i) < p_{\hat{c}} \forall \hat{c} \succ c$) and accepted at school c ($e_c(i) \geq p_c$). Then, adding up over all possible ranking positions k, all preference orderings $\succ \in P_{k,c}$ with positive measure ($F_i(\succ) > 0$), and aggregating over all student types $i \in I$, we get the total demand for school c.

The following result is useful to derive our efficiency bounds.

Lemma 1. *Let p be a market-clearing cutoff vector characterizing a stable matching. Then, for all c, $\phi_c \leq p_c \leq \Phi_c$.*

Proof. For any $x \in [0,1]^N$, $\Lambda_c^1(x_c) \leq D_c(x) \leq \bar{\Lambda}(x_c)$ which are all decreasing in x_c. Then, fix any p_{-c} and let ϕ_c, p_c, Φ_c be solutions to $\Lambda_c^1(\phi_c) = k_c$, $D_c(p_c, p_{-c}) = k_c$ and $\bar{\Lambda}_c(\Phi_c) = k_c$ respectively, it is true that $\phi_c \leq p_c \leq \Phi_c$. □

A.1 Upper bound

Let p be a cutoff vector for a stable matching. Then

$$\begin{aligned}
k_c &= D_c(p)\\
&= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) + \sum_{k=2}^{N}\sum_{\succ\in P_{k,c}} \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}}\forall \hat{c}\succ c} F_i(\succ)\nu(di)\\
&\geq \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) + \sum_{k=2}^{N}\sum_{\succ\in P_{k,c}} \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}}\forall \hat{c}\in \text{supp}(i)\setminus\{c\}} F_i(\succ)\nu(di)\\
&= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) + \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}}\forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \sum_{k=2}^{N}\sum_{\succ\in P_{k,c}} F_i(\succ)\nu(di)\\
&= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) + \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}}\forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \sum_{k=2}^{N} F_i^k(c)\nu(di)\\
&= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) + \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}}\forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di)\\
&:= \Lambda(p_c).
\end{aligned}$$

To see the inequality above, note that for any $k = 2, \ldots, N$, $\succ\in P_{k,c}$, and $\hat{c} \succ c$, it follows that $\hat{c} \in \text{supp}(i) \setminus \{c\}$. Thus, for any $k = 2, \ldots, N$ and $\succ\in P_{k,c}$

$$\Big\{i \in I \mid e_c(i) \geq p_c, e_{\hat{c}}(i) < p_{\hat{c}}\forall \hat{c} \succ c\Big\} \supseteq \Big\{i \in I \mid e_c(i) \geq p_c, e_{\hat{c}}(i) < p_{\hat{c}}\forall \hat{c} \in \text{supp}(i) \setminus \{c\}\Big\}$$

and therefore

$$\int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}}\forall \hat{c}\succ c} F_i(\succ)\nu(di) \geq \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}}\forall \hat{c}\in \text{supp}(i)\setminus\{c\}} F_i(\succ)\nu(di)$$

Since $\bar{\Lambda}(\Phi_c) = k_c \geq \Lambda(p_c)$

$$0 \leq \bar{\Lambda}(\Phi_c) - \Lambda(p_c) = \bar{\Lambda}(p_c) + \int_{p_c}^{\Phi_c} \bar{\Lambda}'(s)ds - \Lambda(p_c) \leq (\Phi_c - p_c) \sup_{x\in[\phi_c,\Phi_c]} \bar{\Lambda}' + \bar{\Lambda}(p_c) - \Lambda(p_c)$$

therefore

$$
\begin{aligned}
&(\Phi_c - p_c)\Big(- \sup \bar{\Lambda}'\Big) \\
&\qquad \leq \bar{\Lambda}(p_c) - \Lambda(p_c) \\
&\qquad = \int_{e_c(i)\geq p_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq p_c, e_{\hat{c}}(i) < p_{\hat{c}} \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di) \\
&\qquad = \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)\geq p_{\hat{c}} \text{ for some } \hat{c}\in \text{supp}(i)\setminus\{c\}} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\qquad \leq \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)\geq \phi_{\hat{c}} \text{ for some } \hat{c}\in \text{supp}(i)\setminus\{c\}} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\qquad = \int_{e_c(i)\geq \phi_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i) < \phi_{\hat{c}} \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di).
\end{aligned}
$$

Since $\sup \bar{\Lambda}' < 0$, it follows that

$$
\begin{aligned}
&(\Phi_c - p_c) \\
&\leq \frac{1}{\big(- \sup \bar{\Lambda}'\big)} \Big(\int_{e_c(i)\geq \phi_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i) < \phi_{\hat{c}} \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di)\Big)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
R^1(c) &= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) \\
&= \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) \quad + \int_{p_c \leq e_c(i)\leq \Phi_c} F_i^1(c)\nu(di) \\
&\leq \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) \quad + \frac{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(-\int_{e_c(i)\geq x} F_i^1(c)\nu(di))}{\big(- \sup \bar{\Lambda}'\big)} \Bigg(\int_{e_c(i)\geq \phi_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\qquad\qquad - \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i) < \phi_{\hat{c}}, \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di)\Bigg)
\end{aligned}
$$

where the inequality follows since

$$
\int_{p_c \leq e_c(i)\leq \Phi_c} F_i^1(c)\nu(di) = \Lambda^1(p_c) - \Lambda^1(\Phi_c) \leq (\Phi_c - p_c) \sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(-\Lambda^1(x)).
$$

It follows that

$$
\begin{aligned}
R^1(c) &\leq \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) + \eta_c \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)>\phi_{\hat{c}} \text{ some } \hat{c}\in\text{supp}(i)\setminus\{c\}} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\leq \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) + \eta_c \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)>\phi_{\hat{c}} \text{ some } \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di)
\end{aligned}
$$

A.2 Lower bound

Define

$$
\hat{\Lambda}_c(x) = \Lambda_c^1(x) + \int_{e_c(i)\geq x, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di)
$$

and note that

$$
\hat{\Lambda}_c(p_c) \geq D_c(p) = k_c
$$

Since $\Lambda_c^1(\phi_c) = k_c$,

$$
\Lambda_c^1(\phi_c) \leq \hat{\Lambda}_c(p_c) \leq \hat{\Lambda}_c(\phi_c) + (p_c - \phi_c) \sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x)
$$

Rearranging terms,

$$
p_c - \phi_c \leq \frac{-1}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x)} \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di)
$$

Now,

$$
\begin{aligned}
R^1(c) &= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) \\
&= \int_{e_c(i)\geq \phi_c} F_i^1(c)\nu(di) - \int_{\phi_c\leq e_c(i)\leq p_c} F_i^1(c)\nu(di) \\
&\geq k_c - (p_c - \phi_c) \sup_{x\in[\phi_c,\Phi_c]} -\frac{d}{dx}\Lambda_c^1(x) \\
&\geq k_c - \frac{\sup_{x\in[\phi_c,\Phi_c]} -\frac{d}{dx}\Lambda_c^1(x)}{-\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x)} \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di)
\end{aligned}
$$

Note that

$$R^1(c) \geq k_c - \sum_{k=2}^{N} \sum_{\succ \in P_{k,c}} \int_{e_c(i) \geq \phi_c, e_{\hat{c}}(i) < \Phi_c \ \ \forall \hat{c} \succ c} F_i(\succ)\nu(di)$$

$$\geq k_c - \sum_{k=2}^{N} \sum_{\succ \in P_{k,c}} \int_{e_c(i) \geq \phi_c, e_{\hat{c}}(i) < \Phi_c \ \ \text{some } \hat{c} \succ c} F_i(\succ)\nu(di)$$

$$= k_c - \int_{e_c(i) \geq \phi_c, e_{\hat{c}}(i) < \Phi_c \ \ \text{some } \hat{c} \succ c} (\bar{F}_i(c) - F_i^1(c))\nu(di),$$

Setting

$$\bar{\eta}_c = \min\left\{1, \frac{\sup_{x \in [\phi_c, \Phi_c]} -\frac{d}{dx}\Lambda_c^1(x)}{-\sup_{x \in [\phi_c, \Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x)}\right\}$$

it follows that

$$R^1(c) \geq k_c - \bar{\eta}_c \int_{e_c(i) \geq \phi_c, e_{\hat{c}}(i) < \Phi_c \ \ \text{some } \hat{c} \in \text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di)$$

B Proofs for Section 4

B.1 Applying Theorem 1 to random and distance-based priorities

We apply Theorem 1 for each priority rule.

B.1.1 Multiple tie breaking

Recall that in this setting $I = [0,1]^N$ and ν are N independent uniform distributions. First, we can specify $\Lambda_c^1(x)$ and $\bar{\Lambda}_c(x)$:

$$\Lambda_c^1(x) = \int_{e_c(i) \geq x} F_i^1(c)\nu(di) = \int \left[\int_x^1 F_i^1(c)du\right]\nu(di) = \int F_i^1(c)\nu(di)\int_x^1 du = F^1(c)(1-x)$$

$$\bar{\Lambda}_c(x) = \int_{e_c(i) \geq x} \bar{F}_i(c)\nu(di) = \int \left[\int_x^1 \bar{F}_i(c)du\right]\nu(di) = \int \bar{F}_i(c)\nu(di)\int_x^1 du = \bar{F}_i(c)(1-x)$$

where the first equality obviates the $N-1$ integrals of measure 1. Therefore

$$\phi_c = 1 - \frac{k_c}{F^1(c)} \qquad \Phi_c = 1 - \frac{k_c}{\bar{F}(c)}$$

Similarly,

$$\begin{aligned}
\Lambda_c(x) &= \Lambda_c^1(x) + \int_{e_c(i)\geq x, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ for some } \hat{c}\in \text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&= F^1(c)(1-x) + \int_{e_c(i)\geq x} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq x, e_{\hat{c}}(i)\geq \Phi_{\hat{c}} \forall \hat{c}\in \text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&= F^1(c)(1-x) + \int \left[\int_x^1 (\bar{F}_i(c) - F_i^1(c))du\right]\nu(di) - \int \left[\int_x^1 \int_{\Phi_{\hat{c}}, \forall \hat{c}\neq c}^1 (\bar{F}_i(c) - F_i^1(c))du\right]\nu(di) \\
&= F^1(c)(1-x) + (\bar{F}(c) - F^1(c))(1-x) - (\bar{F}(c) - F^1(c))(1-x)\prod_{\hat{c}\neq c}(1-\Phi_{\hat{c}}) \\
&= F^1(c)(1-x) + (\bar{F}(c) - F^1(c))(1-x)\left[1 - \prod_{\hat{c}\neq c}\frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]
\end{aligned}$$

Having calculated, $\Lambda_c^1(x), \bar{\Lambda}_c(x), \Lambda_c(x)$, we have that

$$\begin{aligned}
\bar{\eta}_c &= \min\left\{1, \frac{\inf_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\Lambda_c^1(x))}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\Lambda_c(x))}\right\} \\
&= \min\left\{1, \frac{\inf_{x\in[\phi_c,\Phi_c]} -F^1(c)}{\sup_{x\in[\phi_c,\Phi_c]} -F^1(c) - (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c}\frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}\right\} \\
&= \frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c}\frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}
\end{aligned}$$

and

$$\begin{aligned}
\eta_c &= \min\frac{\inf_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\Lambda_c^1(x))}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\bar{\Lambda}_c(x))} = \frac{\inf_{x\in[\phi_c,\Phi_c]} -F^1(c)}{\sup_{x\in[\phi_c,\Phi_c]} -\bar{F}(c)} \\
&= \frac{F^1(c)}{\bar{F}(c)}
\end{aligned}$$

We now calculate our bounds

$$\begin{aligned}
R^1(c) &\geq k_c - \frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c}\frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}\frac{k_c}{F^1(c)}(\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c}\frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right] \\
&= k_c\left(\frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c}\frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}\right)
\end{aligned}$$

and

$$
\begin{aligned}
R^1(c) &\leq F^1(c)\frac{k_c}{\bar{F}(c)} + \frac{F^1(c)}{\bar{F}(c)}(\bar{F}(c) - F^1(c))\frac{k_c}{F^1(c)}\left[1 - \prod_{\hat{c}\neq c}\left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right] \\
&= \frac{k_c}{\bar{F}(c)}\left[F^1(c) + \frac{\bar{F}(c) - F^1(c)}{\bar{F}(c)}\left[1 - \prod_{\hat{c}\neq c}\left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right]\right] \\
&= k_c\left[1 - \frac{\bar{F}(c) - F^1(c)}{\bar{F}(c)}\prod_{\hat{c}\neq c}\left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right]
\end{aligned}
$$

B.1.2 Single tie breaking

Recall that in this case $I = [0,1]$ and ν is given by the uniform distribution. We first note that for single tie breaking, $\phi_c = 1 - \frac{k_c}{F^1(c)}$ and $\Phi_c = 1 - \frac{k_c}{\bar{F}(c)}$. When $\frac{k_c}{F^1(c)} < \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}$ for all $\hat{c} \neq c$, then $\phi_c \geq \Phi_{\hat{c}}$ for all $\hat{c} \neq c$ and therefore

$$
\Big(\int_{e_c(i)\geq\phi_c^{STB}, e_{\hat{c}}(i)<\Phi_{\hat{c}}^{STB} \text{ for some } \hat{c}\neq c}(1 - F_i^1(c))\nu(di)\Big) = 0
$$

Thus, Theorem 1 implies that when $\frac{k_c}{F^1(c)} < \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}$ for all $\hat{c} \neq c$,

$$
R^1(c) = k_c.
$$

Consider now the case in which $\frac{k_c}{F^1(c)} > \min\{\frac{k_{\hat{c}}}{\bar{F}(\hat{c})} \mid \hat{c} \neq c\}$. This condition implies that $\phi_c < \max\{\Phi_{\hat{c}} \mid \hat{c} \neq c\}$. Since $\bar{\eta}_c \leq 1$, we deduce that

$$
\begin{aligned}
R^1(c) &\geq k_c - \bar{\eta}_c\Big(\int_{e_c(i)\geq\phi_c^{STB}, e_{\hat{c}}(i)<\Phi_{\hat{c}}^{STB} \text{ for some } \hat{c}\neq c}(\bar{F}(c) - F_i^1(c))\nu(di)\Big) \\
&= k_c - \left[\int_{e_c(i)\geq\phi_c^{STB}}(\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq\phi_c^{STB}, e_{\hat{c}}(i)>\Phi_{\hat{c}}^{STB}\forall\hat{c}\in\text{supp}(c)}(\bar{F}_i(c) - F_i^1(c))\nu(di)\right] \\
&= k_c - \left[(\bar{F}(c) - F^1(c))(1 - \phi_c^{STB}) - (\bar{F}(c) - F^1(c))(1 - \max\{\phi_c^{STB}, (\max_{\hat{c}\neq c}\Phi_{\hat{c}}^{STB})\})\right] \\
&= k_c - (\bar{F}(c) - F^1(c))\left[(\max_{\hat{c}\neq c}\Phi_{\hat{c}}^{STB}) - \phi_c^{STB}\right].
\end{aligned}
$$

B.1.3 Distance-based priorities

Since $\bar{\eta} \leq 1$, Theorem 1 implies that

$$\begin{aligned} R^1(c) &\geq k_c - \bar{\eta}_c\Big(\int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ for some } \hat{c}\neq c} (1-F_i^1(c))\nu(di)\Big) \\ &= k_c - \bar{\eta}_c \int_{d(i,c)\leq 1-\phi_c^{DB}, d(i,\hat{c})>1-\Phi_{\hat{c}}^{DB} \text{ some } \hat{c}\neq c} (1-F_i^1(c))\nu(di) \\ &\geq k_c - \nu(H(c)) \sup_{d(i,c)\leq 1-\phi_c^{DB}} (1-F_i^1(c)). \end{aligned}$$

Under condition (4.8), Theorem 1 implies that

$$\begin{aligned} R^1(c) &\leq \int_{e_c(i)\geq \Phi_c^{DB}} F_i^1(c)\nu(di) + 0 \\ &= \int_{e_c(i)\geq \Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} \bar{F}_i(c)\nu(di) \\ &\leq k_c \sup_{i:e_c(i)\geq \Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} \\ &= k_c \sup_{i:d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} \end{aligned}$$

B.2 Students assigned to top schools and Pareto efficiency

Lemma 2. *a. For any matching μ, $P^2 \leq P$ and*

$$\sum_{c=1}^{N} R^1(c) + P \leq \sum_{c=1}^{N} k_c. \tag{B.1}$$

b. Assume that $\mathbb{P}[c \succ c' \mid i] > 0$ for all $c \neq c'$ and all $i \in I$. When priorities are built from multiple tie breaking and μ is stable, then $P^2 = P$ and (B.1) holds with equality.

c. Assume that $\mathbb{P}[c \succ c' \mid i] > 0$ for all $c \neq c'$ and all $i \in I$. Assume priorities are distance-based, and that conditions (4.11) and (4.8) hold. Then $P^2 = P$ and (B.1) holds with equality.

Proof of Lemma 2. Let μ be any matching and take S^1 as the set of all students assigned to their top schools. Clearly, no subset of S^1 can be Pareto improved. Let

$$S^P = \bigcup_{S'' \text{ can be Pareto improved}} S''$$

and note that $S^P \subseteq \{s \mid s \text{ is assigned by } \mu\}$. It follows that

$$S^1 \cup S^P \subseteq \{s \mid s \text{ is assigned}\} \text{ and } S^1 \cap S^P = \emptyset.$$

As a result,

$$\bar{\nu}(S^1) + \bar{\nu}(S^P) \leq \bar{\nu}(\{s \text{ is assigned}\}).$$

Since $\bar{\nu}(S^1) = \sum_c R^1(c)$, $\bar{\nu}(S^P) = P$ and $\bar{\nu}(\{s \text{ is assigned }\}) \leq \sum_c k_c$, we deduce that

$$\sum_c R^1(c) + P \leq \sum_c k_c.$$

Take now a stable matching under multiple tie breaking. Consider any student s who is assigned to a school that is not her top choice. We will argue that there is a positive measure set S', that contains s, such that S' is part of Pareto-improving pairs. Let $c = \mu(s)$ and consider a school $\hat{c}$ and a set of students $\hat{S}$ assigned to c such that $\hat{S}$ has positive measure and contains s, and all students in $\hat{S}$ prefer $\hat{c}$ over c. Consider the set of all students who prefer c over $\hat{s}$ but only have scores to get admission to $\hat{c}$:

$$\bar{S} = \{s \in S \mid c \succ_s \hat{c}, \quad i_{\hat{c}} \geq p_{\hat{c}}, \quad p_{c'} > i_{c'} \forall c' \neq \hat{c}\}.$$

Clearly,

$$\bar{\nu}(\bar{S}) = \Big(\int \mathbb{P}[c \succ \hat{c} \mid i] \nu(di) \Big) (1 - p_{\hat{c}}) \prod_{c' \neq \hat{c}} p_{c'} > 0$$

Without loss, assume that $\bar{\nu}(\bar{S}) = \bar{\nu}(\hat{S})$.[26] Construct the matching $\bar{\mu}$ by $\bar{\mu}(c') = \mu(c')$ for all $c' \in C \backslash \{c, \hat{c}\}$ and

$$\bar{\mu}(c) = (\mu(c) \cup \bar{S} \setminus \hat{S} \quad \text{and } \bar{\mu}(\hat{c}) = (\mu(\hat{c}) \cup \hat{S} \setminus \bar{S}.$$

[26] If not, scale down the set with the largest measure so that the measures coincide.

It follows that $\bar{\mu}$ is a matching and $S' = \bar{S} \cup \hat{S}$ is part of Pareto-improving pairs. As a result,

$$\{s \text{ is assigned to a school that is not her top}) \leq \bigcup_{\tilde{S} \text{ is part of Pareto-improving pairs}} \tilde{S}$$

and since

$$\bar{\nu}(\{s \text{ is assigned to a school that is not her top}) = \sum_{c}(k_c - R^1(c))$$

it follows that

$$\sum_{c=1}^{N}(k_c - R^1(c)) \leq \bar{\nu}(\bigcup_{\tilde{S} \text{ is part of Pareto-improving pairs}} \tilde{S}) = P^2.$$

We deduce that under multiple-tie breaking, $P = P^2$ and $\sum_{c=1}^{N}(k_c - R^1(c)) = P^2$. The proof for distance-based priorities is analogous. □

The following example shows that under distance-based priorities, inequality (4.12) can be strict when condition (4.8) does not hold.

Example 7. *Suppose that $N = 2$ and $I = [0, 1]$. School c_1 has capacity k_1 and is located in $3/4$, while school c_2 has capacity k_2 and is located in 1. Students find both schools acceptable and for each i, a fraction 1/2 of students prefer c_1 over c_2.*

Under distance-based priorities, we characterize a stable matching such that all students with score above the cutoff p_1 also have score above p_2 for school c_2:

$$\int_{|i-3/4|<1-p_1} \frac{1}{2} di = k_1 \quad \text{and} \quad \int_{i>p_2} di - \int_{|i-3/4|<1-p_1} \frac{1}{2} di = k_2.$$

The first condition is the market clearing condition for school c_1: the demand for school 1 is given by half of the student living within distance p_1 of the schools. The second condition is the market clearing condition for school c_2: the demand for school 2 is given by all students living with distance p_2 of schools 2 minus the fraction of students that get admission to school 1. We can solve for the cutoffs:

$$(1 - p_1) = k_1 \quad 1 - p_2 = k_1 + k_2$$

with $3/4 - k_1 > 1 - (k_1 + k_2)$ (so that students that can be accepted to c_1 can also be accepted to c_2).

For $k_2 > 1/4$ and $k_1 < k_2$,

$$R^1_{DB}(c_1) = k_1 \quad \textit{and} \quad R^1_{DB}(c_2) = \frac{(k_2 - k_1)}{2}.$$

The matching is Pareto-efficient and $P = 0$, but

$$\sum_c R^1(c) + P < \sum_c k_c.$$

B.3 Proofs for Section 4

Proof of Proposition 1. If $\frac{k_c}{F^1(c)} \leq \min_{\hat{c} \neq c} k_{\hat{c}}$, then $R^1_{STB}(c)$ and thus obviously $R^1_{STB}(c) > R^1_{MTB}(c)$. Consider now the case $\frac{k_c}{F^1(c)} > \min_{\hat{c} \neq c} k_{\hat{c}}$. In this case, single tie breaking results in more students assigned to their top schools provided:

$$k_c \left[1 - (1 - F^1(c)) \prod_{\hat{c} \neq c} \left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})} \right) \right] \leq k_c - (1 - F^1(c)) \left[\frac{k_c}{F^1(c)} - \min_{\hat{c} \neq c} \{k_{\hat{c}}\} \right]$$

This condition is equivalent to

$$\frac{k_c}{F^1(c)} \leq \frac{\min_{\hat{c} \neq c} k_{\hat{c}}}{1 - F^1(c) \prod_{\hat{c} \neq c}(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})})}.$$

□

Proof of Proposition 2. From (4.6), we deduce that $R^1_{DB}(c) = k_c$ under the conditions in the statement.

□

Proof of Proposition 3 . We provide a condition such that the lower bound for multiple tie breaking is larger than the upper bound for distance-based priorities:

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} \leq k_c \left(\frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]} \right)$$

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} - \frac{F^1(c)}{\bar{F}(c)} \leq \frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]} - \frac{F^1(c)}{\bar{F}(c)}$$

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} - \frac{F^1(c)}{\bar{F}(c)} \leq \frac{F^1(c)\bar{F}(c) - F^1(c)F^1(c) - F^1(c)(\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}{\bar{F}(c)F^1(c) + \bar{F}(c)(\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}$$

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} - \frac{F^1(c)}{\bar{F}(c)} \leq \frac{\left(1 - \frac{F^1(c)}{\bar{F}(c)}\right) \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}}{1 + \left(\frac{\bar{F}(c)}{F^1(c)} - 1\right) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}.$$

□

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