Multidimensional Apportionment through Discrepancy Theory*

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Deciding how to allocate the seats of a house of representatives is one of the most fundamental problems in the political organization of societies. The underlying goal is to apportion a house of representatives, reflecting the needs of different segments across the population. The idea of *proportionality*, under which a party receives an amount of seats proportional to its votes, is thus at the core of many apportionment systems. Since in general the seats are not divisible, it is necessary to properly formalize the notion of proportionality in an integral setting. The divisor methods provide an answer to this problem, based on appropriately scaling the votes and rounding the result in order to meet the house size. These methods are widely used at national and regional levels in many democracies around the world. The two most common are the Jefferson/D'Hondt method proposed by Thomas Jefferson in 1792, and the Webster/Sainte-Laguë method first proposed by Daniel Webster in 1832.

In their seminal work, Balinski and Demange extended the notion of proportionality and divisor methods to the case in which the apportionment is ruled by two dimensions, studying this extension from an axiomatic and algorithmic point of view [1, 2]. In this setting, an instance is given by an integral matrix (of votes) where the rows typically represent the political parties and the columns represent the regions. We are also given a list of strictly positive integers, called *marginals*, specifying the row and column sums for any feasible biproportional apportionment. Thus the *row* marginals account for the number of seats that have to be allocated to the corresponding party[1], and the *column* marginals correspond to the number of seats a given district should get.[2] The goal is to find a matrix (of seats) satisfying the marginals and keeping proportionality with respect to the votes simultaneously in both dimensions. This notion is captured by a set of *multipliers*, one for each row and column. The biproportional apportionment method is currently used in elections of several cantons in Switzerland.

A distinctive feature of the biproportional method is that the existence (and essential uniqueness) is guaranteed under very natural conditions [2]. However, by design, the biproportional method is

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[1]Generally, these marginals are given in proportion to the number of votes, by applying some divisor method.
[2]Often proportional to the population of the district.

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limited to the case of apportionments ruled by two dimensions. Departing from the two-dimensional case is not only a challenging mathematical question, but also a relevant practical problem. Indeed, as modern societies become more complex, representation of dimensions beyond political affiliation and geography is increasingly demanded. For instance, New Zealand's parliament includes ethnic representation while the recently elected 2021 Chilean Constitutional Convention includes gender balance as a constraint. Another example, mentioned by Demange [4], is the proposed division of three types of "Constituent People" (Bosniacs, Croats and Others) in the Parliament of the Federation of Bosnia and Herzegovina, which led Demange to coin the multidimensional proportional apportionment as a challenging question.

In this paper, we initiate the study of multidimensional proportional apportionment. Our analysis rests on a natural integer program that extends the network flow approach for the 2-dimensional case [5]. Specifically, we prove that the existence of a multidimensional proportional apportionment is fully characterized by the fact that the linear relaxation of this integer program admits an integer optimal solution. We can then establish that, in general, multidimensional proportional apportionments may fail to exist. Furthermore, we show that determining the existence of a proportional apportionment in the multidimensional setting (dimension 3 and higher) is NP-complete. This is in sharp contrast with the polynomially solvable 2-dimensional case [1, 2].

Given that multidimensional proportional apportionments may fail to exist (and are in general hard to compute) we study what happens when we allow for small violations in the marginals. Specifically, we consider the question of whether we can obtain an apportionment satisfying the proportionality condition by allowing to violate the marginals by a certain amount, and whether this can be done efficiently. Our main result provides a positive answer to this question. It asserts that if $(u_1, \ldots, u_d) \in \mathbb{N}^d$ are the target maximum violations in each dimension,[3] a d-dimensional proportional apportionment exists so long as $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$.[4] In dimension 2, with $u_1 = u_2 = 0$ this recovers the existence result of Balinski and Demange [2], while in dimension 3 a violation of one seat in each dimension is enough to guarantee existence. The proof follows through the lens of discrepancy theory, mainly the celebrated Beck-Fiala Theorem [3]. Specifically, we design an iterative rounding algorithm for an appropriate discrepancy problem in hypergraphs that might be of independent interest.

Finally, we test our method for finding a 3-dimensional proportional apportionment in the context of the 2021 Chilean Constitutional Convention election, that sought to elect a convention achieving proportionality across three dimensions: political, geographical and gender. We conclude that our method is significantly more representative than the one used, and our approach is more robust in that small perturbations of the votes translate into only small changes in the house configuration.

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[3] That is, in each dimension $\ell \in \{1, \ldots, d\}$ we allow the marginals to be additively violated by at most u_ℓ.

[4] The result actually requires a mild additional assumption that, for instance, is satisfied if the original vote matrix does not contain zeros.