The Value of Observing the Buyers' Arrival Time in Dynamic Pricing

APPENDIX

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A1. Complementary results

Proposition A1. *The function $\phi(t)$ is lower semi-continuous.*

Proof. We need to show that for all $t_0 \geq 0$, it holds that

$$\liminf_{t \to t_0} \phi(t) \geq \phi(t_0). \tag{A1}$$

First, note that from the definition of $\phi(t)$, we are looking for a value v to set $\phi(t) = v$, i.e., v must be the smallest valuation verifying: $U(t,v) \geq U(t',v)$, for all $t' > t$[1], or equivalently,

$$e^{-\mu t}(v - p(t)) \geq e^{-\mu t'}(v - p(t')),$$

where by isolating v we get

$$v \geq \frac{p(t) - e^{-\mu(t'-t)}p(t')}{1 - e^{-\mu(t'-t)}}.$$

Observe that due to the lower semi-continuity of $p(t)$, the definition of the threshold function ϕ is equivalent to

$$\phi(t) = \sup_{t' > t} \left\{ \frac{p(t) - e^{-\mu(t'-t)}p(t')}{1 - e^{-\mu(t'-t)}} \right\}. \tag{A2}$$

To prove that ϕ is lower semi-continuous we need the following auxiliary result, whose proof follows from the definition of liminf for functions.

Auxiliary lemma. If f and g are functions such that for all $y \geq x$ it holds that $f(x) \geq g(x,y)$, then $\liminf_{x \to x_0} f(x) \geq \liminf_{x \to x_0} g(x,y)$, for all $y \geq x_0$.

[1] Note that the inequality holds for all v setting $t' = t$ and therefore we can restrict the condition for t' strictly greater than t.

From (A2) we have $\phi(t) \geq \frac{p(t)-e^{-\mu(t'-t)}p(t')}{1-e^{-\mu(t'-t)}}$ for all $t' \geq t$, and using the auxiliary lemma it follows that, for all $t' \geq t_0$,

$$\liminf_{t\to t_0} \phi(t) \geq \liminf_{t\to t_0} \frac{p(t)-e^{-\mu(t'-t)}p(t')}{1-e^{-\mu(t'-t)}}.$$

Due to the lower semi-continuity of $p(t)$ and the continuity of the exponential function, and in view of (A1), the right side of this inequality is at least $\frac{p(t_0)-e^{-\mu(t'-t_0)}p(t')}{1-e^{-\mu(t'-t_0)}}$, and therefore,

$$\liminf_{t\to t_0} \phi(t) \geq \frac{p(t_0)-e^{-\mu(t'-t_0)}p(t')}{1-e^{-\mu(t'-t_0)}} \quad \forall t' \geq t_0.$$

Then, $\liminf_{t\to t_0} \phi(t)$ is at least the maximum, over all $t' \geq t_0$, of $\frac{p(t_0)-e^{-\mu(t'-t_0)}p(t')}{1-e^{-\mu(t'-t_0)}}$, which is equal to $\phi(t_0)$. Thus, ϕ is lower semi-continuous in $\mathbb{R}_0^+$. □

Proposition A2. *The Euler-Lagrange equation associated to the problem*

$$\max_p \int_0^{+\infty} I(t,p(t),p'(t),p''(t))\mathrm{d}t$$

is given by

$$f'\left(p(t)-\frac{p'(t)}{\mu}\right)\left(-\frac{p''(t)}{\mu}+p'(t)\right)(-\delta p(t)+p'(t))+f\left(p(t)-\frac{p'(t)}{\mu}\right)[\delta(\delta-\mu)p(t)-2\delta p'(t)+2p''(t)]=0. \tag{A3}$$

Proof. Recall that $I(t,p(t),p'(t),p''(t)) = e^{-\delta t}p(t)\left(-p'(t)+\frac{p''(t)}{\mu}\right)f\left(p(t)-\frac{p'(t)}{\mu}\right)$. We have to check that

$$\frac{d^2}{dt^2}\frac{\partial I}{\partial p''} - \frac{d}{dt}\frac{\partial I}{\partial p\prime} + \frac{\partial I}{\partial p} = 0 \tag{A4}$$

is equivalent to equation (A3).

The first term of the RHS of (A4) is given by

$$\begin{aligned}\frac{d^2}{dt^2}\frac{\partial I}{\partial p''} = & \frac{e^{-\delta t}}{\mu}f\left(p(t)-\frac{p'(t)}{\mu}\right)\left(p''(t)-2\delta p'(t)+\delta^2 p(t)\right)+\\ & \frac{e^{-\delta t}}{\mu}f'\left(p(t)-\frac{p'(t)}{\mu}\right)\left[2(p'(t)-\delta p(t))\left(p'(t)-\frac{p''(t)}{\mu}\right)+p(t)\left(p''(t)-\frac{p'''(t)}{\mu}\right)\right]+\\ & \frac{e^{-\delta t}}{\mu}f''\left(p(t)-\frac{p'(t)}{\mu}\right)p(t)\left(p'(t)-\frac{p''(t)}{\mu}\right)^2.\end{aligned}$$

On the other hand, computing the second term we obtain

$$\begin{aligned}\frac{d}{dt}\frac{\partial I}{\partial p'} = & \frac{e^{-\delta t}}{\mu}f\left(p(t)-\frac{p'(t)}{\mu}\right)(\delta p(t)-p'(t))+\\ & \frac{e^{-\delta t}}{\mu}f'\left(p(t)-\frac{p'(t)}{\mu}\right)\left[\left(\frac{p''(t)}{\mu}-p'(t)\right)\left(\frac{\delta p(t)-p'(t)}{\mu}+p(t)\right)+\frac{p(t)}{\mu}\left(p''(t)-\frac{p'''(t)}{\mu}\right)\right]+\\ & \frac{e^{-\delta t}}{\mu}f''\left(p(t)-\frac{p'(t)}{\mu}\right)p(t)\left(\frac{p''(t)}{\mu}-p(t)\right)^2.\end{aligned}$$

Finally, the partial derivative of I with respect to p is the following

$$\frac{\partial I}{\partial p} = e^{-\delta t}\left(\frac{p''(t)}{\mu} - p'(t)\right)\left(f\left(p(t) - \frac{p'(t)}{\mu}\right) + p(t) f'\left(p(t) - \frac{p'(t)}{\mu}\right)\right).$$

Hence, equation (A3) comes from using the expressions above and equalizing the LHS of (A4) to zero. □

A2. Proofs of results in the main body

A2.1. Proof of Proposition 1

Let $p(t)$ be an optimal solution of the relaxed problem $[SPO_0^r]$ and suppose that there exists t such that $\psi(t)$ is an inner local maximum. Then, it must hold that

$$\psi'(t) = p'(t) - \frac{p''(t)}{\mu} = 0. \tag{A5}$$

Recalling that the valuation density f is positive, observe that, at t, the Euler-Lagrange equation (3) becomes

$$\delta(\delta-\mu)p(t) - 2\delta p'(t) + 2p''(t) = 0,$$

and therefore, together with (A5), $p'(t) = \frac{\delta p(t)}{2}$, and thus, $p'(t) - \delta p(t) < 0$.

Let $\epsilon > 0$ and $\rho > 0$ be such that $t_1 = t - \epsilon$ and $t_2 = t + \rho$ satisfying $\psi(t_1) = \psi(t_2)$ and $p'(t_i) - \delta p(t_i) < 0$ for $i = 1, 2$. Furthermore, since ψ has a maximum at t, it must hold that $\psi'(t_1) > 0$ and $\psi'(t_2) < 0$.

Let us first suppose that $f'(\psi(t_1)) = f'(\psi(t_2)) \geq 0$. In this case, considering the first term in (3),

$$\underbrace{f'\left(p(t_2) - \frac{p'(t_2)}{\mu}\right)}_{f'(\psi(t_2)\geq 0}\underbrace{\left(-\frac{p''(t_2)}{\mu} + p'(t_2)\right)}_{\psi'(t_2)<0}\underbrace{(-\delta p(t_2) + p'(t_2))}_{<0} \geq 0,$$

and therefore, since $p(t)$ satisfies the Euler-Lagrange equation (3) for all t –and, in particular, for t_2–, we must have

$$\delta(\delta-\mu)p(t_2) - 2\delta p'(t_2) + 2p''(t_2) \leq 0. \tag{A6}$$

Since by construction $p'(t_2) - \delta p(t_2) < 0$, we have $\frac{p'(t_2)}{\delta} < p(t_2)$, and bounding from below the first term in the LHS of (A6), we obtain

$$-(\delta+\mu)p'(t_2) + 2p''(t_2) < 0. \tag{A7}$$

Recalling that $t_2 = t + \rho$, taking the liminf in the LHS of (A7) when $\rho \to 0$, by the lower semi-continuity of the price function $p(t)$, we obtain

$$-(\delta+\mu)p'(t) + 2p''(t) \leq 0,$$

which is equivalent to $2p''(t) \leq (\delta+\mu)p'(t)$. But from (A5), $\mu p'(t) = p''(t)$, so $2\mu p'(t) \leq (\delta+\mu)p'(t)$ and therefore $\mu \leq \delta$ (because $p'(t) = \frac{\delta p(t)}{2} > 0$), which is a contradiction.

Now, consider the case where $f'(\psi(t_1)) = f'(\psi(t_2)) < 0$. Then, it must hold that

$$\underbrace{f'\left(p(t_1) - \frac{p'(t_1)}{\mu}\right)}_{f'(\psi(t_1))<0} \underbrace{\left(-\frac{p''(t_1)}{\mu} + p'(t_1)\right)}_{\psi'(t_1)>0} \underbrace{(-\delta p(t_1) + p'(t_1))}_{<0} > 0$$

and now we can proceed analogously to the argument above.

Therefore $\psi(t)$ cannot have an inner local maximum, and with a similar argument, neither an inner local minimum. Hence, $\psi(t)$ has to be monotone.

We are now left with showing that the function $\psi(t)$ is indeed non increasing. By contradiction, suppose that ψ is increasing. We will see that if so we could improve the expected revenue, contradicting that ψ corresponds to the optimal solution of the relaxed problem $[SPO_0^r]$. To this end, let us consider the constant function $\hat{p}(t) = p(0)$ for all t. Then, $\hat{\psi}(t) = p(0)$ and therefore the value of the objective function of $[SPO_0^r]$ by considering the feasible pricing policy $\hat{p}$ is given by

$$\hat{p}(0)(1 - F(\hat{\psi}(0))) + \int_0^\infty e^{-\delta t}\hat{p}(t)(-\hat{\psi}'(t))f(\hat{\psi}(t))\mathrm{d}t = p(0)(1 - F(p(0))).$$

On the other hand, the expected revenue of the seller under the pricing policy p can be computed as

$$p(0)(1 - F(\psi(0))) + \int_0^\infty e^{-\delta t}p(t)(-\psi'(t))f(\psi(t))\ \mathrm{d}t.$$

Note that the second term is negative and therefore the expression above is upper bounded by the expected revenue obtained by selling at time 0. That is,

$$p(0)(1 - F(\psi(0))) + \int_0^\infty e^{-\delta t}p(t)(-\psi'(t))f(\psi(t))\ \mathrm{d}t < p(0)(1 - F(\psi(0))).$$

Note that $1 - F(\psi(0)) > 1 - F(p(0))$, and therefore the expected revenue under the price function $\hat{p}$ is greater than the expected revenue under the price function p, which contradicts the optimality of p. Thus, we can conclude that ψ is a non increasing function.

A2.2. Proof of Proposition 2

Given a pair $(p(t), \psi(t))$ solution of $[SPO_0^r]$, with $\psi(t) = p(t) - \frac{p'(t)}{\mu}$ for all t, we must show that it meets the equilibrium constraint of $[SPO_0]$, that is:

$$t \in \arg\max_{s\geq 0}\ e^{-\mu s}(\psi(t) - p(s))\quad \forall t. \tag{A8}$$

Let $h(s) = e^{-\mu s}(\psi(t) - p(s))$, leading to

$$h'(s) = e^{-\mu s}(-\mu(\psi(t) - p(s)) - p'(s)),$$

and

$$h''(s) = -\mu e^{-\mu s}(-\mu(\psi(t) - p(s)) - p'(s)) + e^{-\mu s}(\mu p'(s) - p''(s)).$$

Given an interior solution t of (A8), it must verify $h'(t) = 0$ and

$$h''(t) = \mu e^{-\mu t}\left(p'(t) - \frac{p''(t)}{\mu}\right) \leq 0.$$

Since $(p(t), \psi(t))$ is solution of $[SPO_0^r]$, then from Proposition 1 we know that $\psi'(t) \leq 0$, and therefore, $h''(t) \leq 0$. Hence, $t \in \arg\max_{s\geq 0}\ e^{-\mu s}(\psi(t) - p(s))$, for any pair of functions $(p(t), \psi(t))$ solution of $[SPO_0^r]$. Recalling that the solution of $[SPO_0^r]$ defines an upper bound of $[SPO_0]$, we have that such pair $(p(t), \psi(t))$ indeed defines a solution to $[SPO_0]$.

A2.3. Proof of Lemma 1

Note that an equivalent inequality would be

$$\begin{aligned} cR_0 &\geq R_0 - \left[p(0)(1 - F(\psi(0))) + \int_0^T e^{-\delta t} p(t) \mathrm{d}(1 - F(\psi(t)))\right] \\ &= \int_T^\infty e^{-\delta t} p(t) \mathrm{d}(1 - F(\psi(t))) \end{aligned}$$

By contradiction, suppose that for $T = \ln(1/c)/\delta$, we have that:

$$\int_T^\infty e^{-\delta t} p(t) \mathrm{d}(1 - F(\psi(t))) > c\left[p(0)(1 - F(\psi(0))) + \int_0^\infty e^{-\delta t} p(t) \mathrm{d}(1 - F(\psi(t)))\right]. \tag{A9}$$

Consider the price function $\hat{p}(t) = p(t+T)$ and its associated purchasing function $\hat{\psi}$. The seller's expected revenue can be computed as:

$$R_{\hat{p}} = \hat{p}(0)(1 - F(\hat{\psi}(0))) + \int_0^\infty e^{-\delta t} \hat{p}(t) \mathrm{d}(1 - F(\hat{\psi}(t))).$$

By the definition of $\hat{p}$ and doing the change of variable $u = t+T$, it follows that the seller's expected revenue is given by:

$$R_{\hat{p}} = p(T)(1 - F(\psi(T))) + e^{\delta T} \int_T^\infty e^{-\delta t} p(t) \mathrm{d}(1 - F(\psi(t))).$$

Applying (A9), it follows that this expression verifies

$$R_{\hat{p}} > p(T)(1-F(\psi(T))) + e^{\delta T} c \left[p(0)(1-F(\psi(0))) + \int_0^\infty e^{-\delta t} p(t) \mathrm{d}(1-F(\psi(t))) \right].$$

Note that $p(T)(1-F(\psi(T)))$ is non negative, and that $T = \ln(1/c)/\delta$ implies $e^{\delta T} c = 1$. Thus, the seller's expected revenue for the pricing policy $\hat{p}$ is bigger than the seller's expected revenue for the pricing policy p, which contradicts the optimality of the price function p.

A2.4. Proof of Proposition 3

Suppose that we have the periodic function $\hat{p}$ with period $2T$ given by (4) and consider a random shift, that is, for a random variable $t_0 \sim \text{Unif}[0, 2T]$, consider the function $\hat{p}_{t_0}(t) = \hat{p}(t + t_0)$. Then, given that the buyer arrives in the interval $I_{2k-1} \cup I_{2k}$ of length $2T$, for some $k \in \mathbb{N}$, and denoting by X the random variable *arrival time*, we have the following:

$$\begin{aligned}\mathbb{P}(X \leq t | X \in (I_{2k-1} \cup I_{2k})) &= \mathbb{P}(X \leq t | X \in (2(k-1)T - t_0, 2kT - t_0]) \\ &= \mathbb{P}(X \in (2(k-1)T - t_0, t]).\end{aligned}$$

Letting s be the length of the interval $[2(k-1)T - t_0, t]$, i.e., $s = t - (2(k-1)T - t_0)$, the expression above verifies

$$\begin{aligned}\mathbb{P}(X \leq t | X \in (2(k-1)T - t_0, 2kT - t_0]) &= \mathbb{P}(X \in (2(k-1)T - t_0, 2(k-1)T - t_0 + s]) \\ &= \mathbb{P}(2(k-1)T - X < t_0 \leq 2(k-1)T - X + s) \\ &= \frac{s}{2T} \quad (\text{because } t_0 \sim \text{Unif}[0, 2T]) \\ &= \frac{t - (2(k-1)T - t_0)}{2T},\end{aligned}$$

which proves that X is uniformly distributed in $I_{2k-1} \cup I_{2k}$, and the proof is completed.

A2.5. Proof of Lemma 2

Without loss of generality let us suppose that $k = 1$, that is, the buyer arrives at time $t_0 + t$ belonging to $\tilde{I}_1$ with valuation $v \geq p(T)$, and further assume that he will not purchase before time $T + t_0$ so that the seller is making less revenue than she could really make.

To prove the lemma we analyze the consumer behavior in the unobservable case under the pricing policy $\hat{p}$ depending on his valuation. More specifically we will prove the followings three statements:

1. If $v \in [p(T), \psi(T))$, then the buyer buys at time $2T + t_0$.
2. If $v \in [\psi(T), \psi(0))$, then the buyer waits and buys at time $\tau \in (T + t_0, 2T + t_0]$ satisfying $\psi(\tau) = v$.

3. If $v \geq \psi(0)$ the buyer purchases at time $t_0 + T$.

First, consider a buyer with valuation $v \in [p(T), \psi(T))$. Knowing that he will purchase to gain some positive utility (eventually at time $2T + t_0$), if he decides to buy at time $\tau < 2T + t_0$, then by the monotonicity of the purchasing function ψ in the observable case, we have that $\psi(\tau - (T + t_0)) > \psi(2T + t_0 - (T + t_0)) = \psi(T)$ and it means that the buyer must have valuation greater than $\psi(T)$ to be optimum to purchase at time τ, which is not the case. We then conclude that in this case he will buy at time $2T + t_0$.

Secondly, if the buyer has valuation $v \in [\psi(T), \psi(0))$, then by using the calculation of the purchasing function for the observable arrival case –conducted under the assumption that the buyer arrives at time 0–, we have that for some $t \in [0, T]$, it holds that $v = \psi(t)$, i.e.,

$$t \in \arg\max_{s \geq 0} U(s, \psi(t)),$$

which means that

$$e^{-\mu t}(\psi(t) - p(t)) \geq e^{-\mu s}(\psi(t) - p(s)), \ \forall s \geq 0.$$

This is equivalent to

$$e^{-\mu(T+t_0)} e^{-\mu t}(\psi(t) - p(t)) \geq e^{-\mu(T+t_0)} e^{-\mu s}(\psi(t) - p(s)), \ \forall s \geq 0.$$

Hence, the buyer will buy at time $\tau = T + t_0 + t$ satisfying $\psi(t) = v$.

Finally, the third statement follows directly from the definition of the threshold function ψ.

The lemma follows by observing that if the buyer has valuation at least $\psi(T)$, the seller's revenue is the same as in the observable case with the buyer arriving at time $T + t_0$ and accumulating revenue up to time $2T + t_0$ (cases (2) and (3)). But if the buyer has valuation between $p(T)$ and $\psi(T)$ (case (1)), then he will buy before time $2T + t_0$ in the unobservable setting under the price function $\hat{p}$, but he will buy after that time in the observable case with arrival time $T + t_0$.

Therefore, we conclude that, conditioned on the event that the buyer with valuation greater than $p(T)$ arrives at time $T + t_0$ –which is equivalent to looking at the problem in the interval $[T + t_0, 2T + t_0]$ in the observable case–, the seller's expected revenue under the policy $\hat{p}$ in the unobservable case is at least the expected revenue earned up to time $2T + t_0$ in the observable case with arrival time $T + t_0$.

A2.6. Proof of Lemma 3

Consider the pricing policy $\hat{p}$ described in Figure 2 in Section 5 and fix the buyer arrival time τ. Recall that t_0 is the uniform random variable involved in the random shift applied over the original price function p to get $\hat{p}$. Recall that these functions have period $2T$, where we are setting $T = \ln(1/c)/\delta$.

Suppose, without loss of generality, that the buyer arrives during the first cycle of the policy; i.e., $\tau \in [t_0, t_0 + 2T]$. Thus, $t_0 \sim \text{Unif}[\tau - 2T, \tau]$. In order to have intervals defined around t_0, we denote $\tilde{\mathcal{I}}_1 := [\tau - T, \tau]$ and $\tilde{\mathcal{I}}_2 := [\tau - 2T, \tau - T]$. With this definition, we have that $\tau \in \tilde{I}_i$ if and only if $t_0 \in \tilde{\mathcal{I}}_i$, for $i = 1, 2$.

In our analysis we will only consider the buyer's arrival if it belongs to the interval $\tilde{I}_1$, otherwise, we simply bound the revenue by 0.

Note that if $\tau \in \tilde{I}_1$, we can lower bound R_τ^{uo} by the expected revenue obtained by considering that the buyer has valuation at least $p(T)$ and that he purchases after time $t_0 + T$. This is because the buyer does not purchase if $v < p(T)$, and his wait until $t_0 + T$ to buy when he could have bought earlier would only hurt the seller's revenue. Then, from Lemma 2, R_τ^{uo} is at least the expected revenue earned up to time $2T + t_0$ in the observable case with arrival time $T + t_0$, i.e., $R_\tau^{uo} \geq R_{[t_0+T, t_0+2T]}$.

Let R_{t_0+T} be the expected revenue in the observable case if the buyer arrives at time $t_0 + T$, for a given value t_0. After applying Lemma 1, we have $R_{[t_0+T,t_0+2T]} \geq (1-c)R_{t_0+T}$, so that $R_\tau^{uo} \geq (1-c)R_{t_0+T}$.

We now use the analysis above to compute a bound for the expected value of the seller's revenue in the unobservable case conditioned on the event that the buyer arrives at time τ. To this end, we define the sample path-based revenue S_τ^{uo} for the unobservable case from time τ onward, i.e., $R_\tau^{uo} = \mathbb{E}(S_\tau^{uo})$, and conditioning on the random, shifted origin time t_0, we get:

$$
\begin{aligned}
\mathbb{E}(S_\tau^{uo}) &= \mathbb{E}_{t_0}(\mathbb{E}(S_\tau^{uo} \mid t_0)) \\
&= \mathbb{E}(S_\tau^{uo} \mid t_0 \in \tilde{\mathcal{I}}_1)\mathbb{P}(t_0 \in \tilde{\mathcal{I}}_1) + \mathbb{E}(S_\tau^{uo} \mid t_0 \in \tilde{\mathcal{I}}_2)\mathbb{P}(t_0 \in \tilde{\mathcal{I}}_2) \\
&= \frac{1}{2}\mathbb{E}(S_\tau^{uo} \mid t_0 \in \tilde{\mathcal{I}}_1) + \frac{1}{2}\mathbb{E}(S_\tau^{uo} \mid t_0 \in \tilde{\mathcal{I}}_2) \\
&\geq \frac{1}{2}(1-c)\mathbb{E}_{t_0}(R_{t_0+T} \mid t_0 \in \tilde{\mathcal{I}}_1),
\end{aligned}
$$

where the last equality holds because $t_0 \sim \text{Unif}[\tau - 2T, \tau]$, and the inequality follows from the observation above.

Note that $R_{t_0+T} = e^{-\delta(t_0+T-\tau)}R_\tau = ce^{-\delta(t_0-\tau)}R_\tau$, where the second equality holds from $e^{-\delta T} = c$. Therefore, it is enough to compute $\mathbb{E}_{t_0}\left(e^{-\delta(t_0-\tau)} \mid t_0 \in \tilde{\mathcal{I}}_1\right)$. In fact, now for $t_0 \sim \text{Unif}[\tau - T, \tau]$, we have

$$
\begin{aligned}
\mathbb{E}_{t_0}\left(e^{-\delta(t_0-\tau)} \mid t_0 \in \tilde{\mathcal{I}}_1\right) &= \int_{\tau-T}^{\tau} e^{-\delta(t_0-\tau)}\frac{1}{T}\mathrm{d}t_0 \\
&= \frac{e^{\delta T} - 1}{\delta T}.
\end{aligned}
$$

By the definition of T, we know that $T\delta = \ln(1/c)$ and $e^{\delta T} = 1/c$, and therefore we have

$$
\mathbb{E}_{t_0}\left(e^{-\delta(t_0-\tau)} \mid t_0 \in \tilde{\mathcal{I}}_1\right) = \frac{1-c}{c\ln(1/c)}.
$$

We then obtain the following lower bound for the expectation of the seller's revenue in the unobservable case that depends on c:

$$R_\tau^{uo} = \mathbb{E}(S_\tau^{uo}) \geq \frac{(1-c)^2}{2\ln(1/c)} R_\tau,$$

which completes the proof.

A3. Discussion of model assumptions

Our model is quite general and follows the standard setup of the OM literature. Yet, there a few features that are worth discussing.

First, we consider a consumer's utility function of the intertemporal type (e.g., see chapter 20 in Mas-Colell et al. (1995)), where the buyer discounts the net payoff $v - p(t)$ if he decides to buy at a later time t, following the classic econ approach (e.g., see Landsberger and Meilijson (1985), Besanko and Winston (1990)). It has also been the prevailing utility function within the OM literature on strategic consumer behavior (Gönsch et al. (2013)). An alternative model would be one where only the valuation of the buyer declines, obtaining the utility function $U(t,v) = e^{-\mu t}v - p(t)$ (e.g. Aviv and Pazgal (2008), Cachon and Swinney (2009)). However, this model is equivalent to one where a utility function like ours is considered and the seller is more impatient than the buyer. Indeed, by considering the utility function $U(t,v) = e^{-\mu t}v - p(t)$ and a seller discount rate $\delta \geq 0$, the setup is equivalent to defining the utility function $\bar{U}(t,v) = e^{-\mu t}(v - \bar{p}(t))$ and a seller's discount rate $\mu + \delta$, with $\bar{p}(t) = e^{\mu t}p(t)$. In this case, $U(t,v) = e^{-\mu t}v - p(t) = e^{-\mu t}(v - e^{\mu t}p(t)) = e^{-\mu t}(v - \bar{p}(t)) = \bar{U}(t,v)$, and in both cases the seller gets $e^{-\delta t}p(t) = e^{-(\delta+\mu)t}\bar{p}(t)$.

Second, in our model we assume that the seller is more patient than the consumers, i.e., $\delta < \mu$. Otherwise, when $\delta \geq \mu$, the optimal pricing policy is to fix a constant price equal to the monopoly price, obtaining that VO is trivially 1. It is enough to prove it for the observable case because if the optimal price function is constant (and independent of the parameters of the problem), then the advantage of being able to observe the arrival vanishes and both problems are equivalent.

To see this, consider first the case where both discount rates are equal ($\delta = \mu$). This is exactly the setting considered by Stokey (1979) in which she proved that no price discrimination occurs and the optimal pricing policy is to charge the monopoly price, namely p_m, during the whole selling horizon. If the seller is more impatient than the buyer, i.e. $\mu < \delta$, it is enough to note that her optimal expected revenue is upper bounded by the one in the case $\mu = \delta$ since a bigger seller's discount rate can only lead to a worse revenue for her, which is achieved by taking $p(t) = p_m, \forall t$. At the same time, a fixed pricing policy is a feasible solution for the unobservable case, and the revenue derived from it provides a lower bound for R^{uo}. All in all, in this case VO is upper bounded by 1, and hence it is exactly 1.

A4. Solving the observable case

In the particular case when the buyer's valuation has density function $f(x) = kx^{\alpha}$, for some integer number α and positive number k, we can explicitly calculate the pricing function $p(t)$ and the purchasing function $\psi(t)$ for the observable case.

To start with, the problem $[SPO_0^r]$ becomes:

$$\max_p \int_0^{\infty} I(t, p(t), p'(t), p''(t))\mathrm{d}t + p(0)(1 - F(\psi(0))), \tag{A10}$$

where $I(t, p(t), p'(t), p''(t)) = e^{-\delta t} p(t) k \left(p(t) - \frac{p'(t)}{\mu}\right)^{\alpha} \left(-p'(t) + \frac{p''(t)}{\mu}\right)$.

By formulating the Euler-Lagrange equation (3) in this case, we obtain the following ODE in the function $p(t)$ [2]

$$\alpha \left(p'(t) - \frac{p''(t)}{\mu}\right)(p'(t) - \delta p(t)) + \left(p(t) - \frac{p'(t)}{\mu}\right)(\delta(\delta - \mu)p(t) - 2\delta p'(t) + 2p''(t)) = 0,$$

which is equivalent to

$$\delta(\delta - \mu) - \frac{p'(t)}{p(t)}\delta\left(\alpha + 1 + \frac{\delta}{\mu}\right) + \frac{p'(t)^2}{p(t)^2}\left(\alpha + \frac{2\delta}{\mu}\right) - \frac{p'(t)p''(t)}{p(t)^2}\frac{1}{\mu}(2 + \alpha) + \frac{p''(t)}{p(t)}\left(\frac{\alpha\delta}{\mu} + 2\right). \tag{A11}$$

Setting $y(t) = (\log(p(t)))'$, we have that $p'(t)/p(t) = y(t)$, $p'(t)^2/p(t)^2 = y^2(t)$, $p'(t)p''(t)/p(t)^2 = y(t)(y'(t) + y^2(t))$, and $p''(t)/p(t) = (y'(t) + y^2(t))$, which allows to rewrite equation (A11) as:

$$\delta(\delta - \mu) - y(t)\delta\left(\alpha + 1 + \frac{\delta}{\mu}\right) + y^2(t)\left(\alpha + \frac{2\delta}{\mu}\right) - (y(t)y'(t) + y^3(t))\frac{1}{\mu}(2 + \alpha) + (y'(t) + y^2(t))\left(\frac{\alpha\delta}{\mu} + 2\right) = 0.$$

Rearranging terms, this equation can be written as

$$\begin{aligned} &\delta(\delta - \mu) - y(t)\delta\left(\alpha + 1 + \frac{\delta}{\mu}\right) + y^2(t)(2 + \alpha)\left(1 + \frac{\delta}{\mu}\right) \\ &-y^3(t)\frac{1}{\mu}(2 + \alpha) - y(t)y'(t)\frac{1}{\mu}(2 + \alpha) + y'(t)\left(\frac{\alpha\delta}{\mu} + 2\right) = 0. \end{aligned} \tag{A12}$$

Note that (A12) is a separable first-order nonlinear ODE, that is, can be written as $y' = H(y)$, for some function H. Therefore it can be solved by integration. For our case the solution turns out to be quite contrived (and obtainable only in impliocit form), but for two special cases we can give clean closed form solutions. We thus show the optimal pricing policy for the cases where the buyer's valuation distribution is uniform and truncated Pareto.

[2] We highlight here that the equation gives the optimal solution of (A10) for $p(t) - p'(t)/\mu$ in the support of the valuation distribution. Otherwise, the equation is reduced to $0 = 0$.

Case $\alpha=0$. We first fix $k=1$ and $\alpha=0$, i.e., we consider the buyer's valuation Unif[0,1]. In this case, we obtain a second order ODE in the function $p(t)$ with constant coefficients, expressed by

$$p''(t)-\delta p'(t)+\frac{\delta^2-\delta\mu}{2}p(t)=0.$$

Its solution is given by:

$$p(t)=c_1 e^{\frac{1}{2}t(\delta-\sqrt{-\delta(\delta-2\mu)})}+c_2 e^{\frac{1}{2}t(\delta+\sqrt{-\delta(\delta-2\mu)})},$$

where c_1, c_2 are constants to be determined.

Note that $\delta+\sqrt{-\delta(\delta-2\mu)}>0$ and $\delta-\sqrt{-\delta(\delta-2\mu)}<0$ due to $\mu>\delta$. Therefore, the optimal pricing function is a sum of a negative exponential function and a positive exponential function and $p(t)$ could in principle go to infinity when t goes to infinity. However, $p(t)\in[0,1]$ for all t, and then, it must be the case that $c_2=0$. Thus, the optimal price function is a negative exponential function of the form:

$$p(t)=c_1 e^{\frac{1}{2}t(\delta-\sqrt{-\delta(\delta-2\mu)})},$$

with $c_1=p(0)$.

In order to simplify the notation, define the positive constant $A=-\delta+\sqrt{-\delta(\delta-2\mu)}$. We are left with finding the value $p(0)$. Replacing the function $p(t)$ in the unconstrained problem (A10), we can rewrite it as a maximization problem over $p(0)$ as follows:

$$\max_{p(0)}\left\{p^2(0)\int_0^\infty e^{-(\delta+A)t}\left(\frac{A}{2}+\frac{A^2}{4\mu}\right)\mathrm{d}t+p(0)\left(1-p(0)-p(0)\frac{A}{2\mu}\right)\right\}.$$

Solving this problem, we obtain $p(0)=\frac{2\mu(\delta+A)}{(A+2\mu)(A+2\delta)}$. Noting that $p'(t)=-\frac{1}{2}Ap(0)e^{-\frac{1}{2}At}$, we also obtain $p'(0)=-\frac{A\mu(\delta+A)}{(A+2\mu)(A+2\delta)}$. Therefore, the pricing function that solves the seller's problem is given by

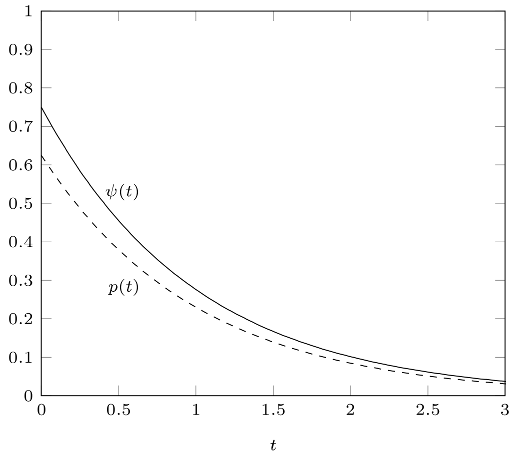
$$p(t)=\frac{2\mu(\delta+A)}{(A+2\mu)(A+2\delta)}e^{-\frac{A}{2}t},$$

with corresponding purchasing function (derived from (1))

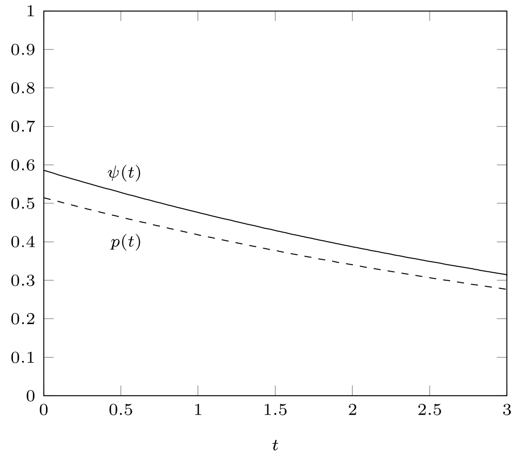
$$\psi(t)=\frac{\delta+A}{A+2\delta}e^{-\frac{A}{2}t}.$$

Note that in this uniform valuation case, the purchasing function turns out to be a positive multiplicative scaling of the pricing function. The optimal expected revenue of the seller in this case can be easily computed obtaining

$$R=\frac{\mu(A+\delta)}{(A+2\mu)(A+2\delta)}.$$



(a) Very impatient buyer ($\mu = 5$). Expected revenue: 0.3125.



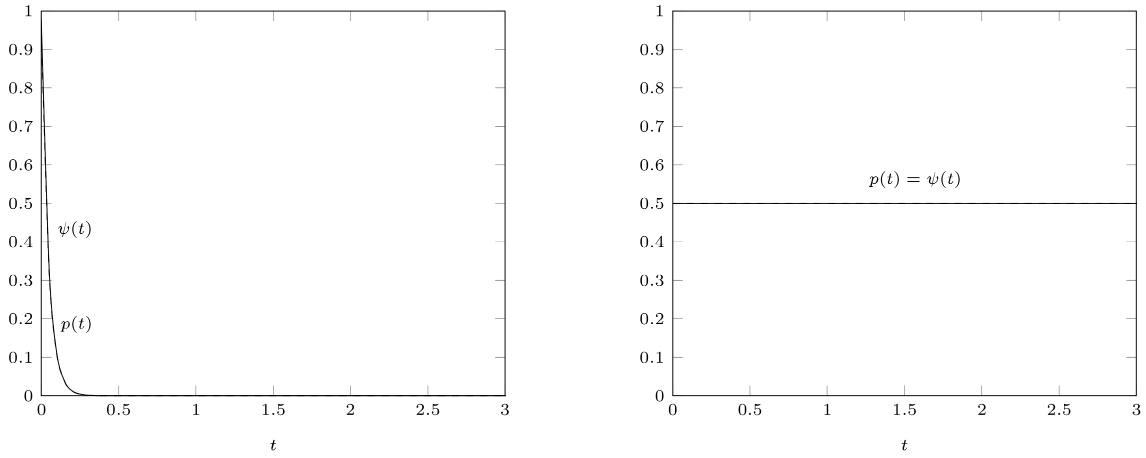
(b) Slightly impatient buyer ($\mu = 1.5$). Expected revenue: 0.2574.

Figure A1 Optimal purchasing and price functions for different levels of asymmetry in the patience of the seller and the buyer. In both panels we normalize the discount rate of the seller at $\delta = 1$.

In what follows, we analyze the optimal curves obtained for some specific values of the discount rates μ and δ, corresponding to different levels of asymmetry in the patience of the seller and the buyer. Without loss of generality, we normalize the discount rate of the seller by setting $\delta = 1$.

In Figure A1, the left panel captures the case where the buyer is five times more impatient than the seller, whereas the right panel illustrates the scenario where he is only 50% more impatient. In panel (a), when the buyer is noticeably more impatient, we can observe that the optimal initial values of $p(0)$ and $p'(0)$ are greater than in panel (b), and that both price and purchasing optimal functions decrease faster. These curves reflect the fact that when facing a more impatient consumer (panel (a)), the seller will price more aggressively early in the horizon but will also drop the price relatively fast. Noting that the decreasing price pattern plays the role of a valuation discovery mechanism, the wider span of the pricing in (a) attempts to keep in the market a low valuation consumer by offering an attractive enough price relatively soon. On the contrary, when the buyer is more patient (panel (b)), the seller can offer a slow decaying price curve so that a consumer with mid to low valuation will buy later (compared to (a)) but at a higher price. The fact that the seller takes advantage of the buyer's impatience is confirmed when computing the ex-ante expected revenue by solving $[SPO_0]$ in both cases, leading to values 0.3125 and 0.2574, respectively.

Figure A2 illustrates two limit scenarios for a normalized seller's discount rate $\delta = 1$. In panel (a) we consider the case in which the buyer is extremely impatient (with $\mu = 1000$). Here, the seller drops the price very quickly from 1 to 0, charging almost instantaneously the valuation of the buyer and extracting his whole surplus. In panel (b) we present the case in which the buyer's discount rate



(a) Extremely impatient buyer ($\mu = 1000$). Expected revenue: 0.4786.

(b) Very patient buyer ($\mu \to \delta$). Expected revenue: 0.25.

Figure A2 Optimal purchasing and price functions for limiting asymmetries in the patience level of the seller and the buyer for different levels of asymmetry in the patience of the seller and the buyer. In both panels we normalize the discount rate of the seller at $\delta = 1$.

tends to 1. The optimal price and purchasing functions are the same and equal to 0.5 throughout the selling horizon. In this case, we recover the optimal auction of Myerson (1981), with reservation price 0.5 and the buyer purchasing at time zero if and only if his valuation is at least 0.5. In this case, he pays the reservation price for the item. The seller's advantage revenue-wise is even more emphasized, with values 0.4786 and 0.25, respectively.

Case $\alpha = -2$. We now take $k = M/(M-1)$ and $\alpha = -2$, which corresponds to the truncated Pareto distribution with parameter 1 and support $[1, M]$.

After some algebra and taking $\delta = 1$, we obtain a second order ODE in the function $p(t)$, expressed by

$$2p''(t)p(t) + p'(t)p(t) - \mu p(t)^2 - 2p'(t)^2 = 0,$$

whose solution is given by

$$p(t) = c_2 e^{\mu t + c_1 e^{-t/2}},$$

where c_1, c_2 are constants to be determined.

Therefore, the optimal pricing policy is

$$p^*(t) = \begin{cases} c_2 e^{\mu t + c_1 e^{-t/2}} & \text{if } t \leq \tilde{t} \\ 1 & \text{if } t > \tilde{t}, \end{cases}$$

where $\tilde{t}$ is such that $c_2 e^{\mu t + c_1 e^{-t/2}} = 1$. On the other hand, it must hold that $p^{*\prime}(t) = 0$, and from both conditions we obtain $c_1 = 2\mu e^{\tilde{t}/2}$ and $c_2 = e^{-\mu(\tilde{t}+2)}$.

Replacing the function $p(t)$ in the unconstrained problem (A10), we can rewrite it as a maximization problem over $\tilde{t}$ as follows:

$$\max_{\tilde{t}} \quad \left\{ \frac{1}{M-1} \left(Me^{-\tilde{t}/2} \left(1 + (2\mu - 1)(e^{-\tilde{t}/2} - 1) + \mu e^{\tilde{t}/2}(1 - e^{-\tilde{t}}) \right) - e^{-\mu(\tilde{t}+2) + 2\mu e^{\tilde{t}/2}} \right) \right\}. \tag{A13}$$

We conclude that the optimal price function of the observable case when the buyer's valuation is TruncPareto(1,1,M) is given by

$$p^*(t) = \begin{cases} e^{-\mu(\tilde{t}+2-t) - 2\mu e^{\tilde{t}/2 - t/2}} & \text{if } t \leq \tilde{t} \\ 1 & \text{if } t > \tilde{t}, \end{cases}$$

where $\tilde{t}$ is the optimal solution of problem A13.

A5. Performance of two heuristic solutions of the unobservable case

In this section we evaluate the performance of two heuristics that may be implemented in the context of the difficult unobservable arrival case, for which an exact solution is hard to characterize: the periodic pricing policy presented in Section 5, and the simple fixed price policy.

In particular, we will assume that the buyer arrives according to an exp(1) distribution, and consider two valuation scenarios: Unif[0,1] and TruncPareto(1,1,100) –as defined in Section 6.2. We normalize the seller's discount rate to 1, and vary the buyer's discount rate μ starting from values close to 1.

In Table A1 we present the results for the uniform valuation case. More specifically, for different values of μ listed in the first column, we use the analysis in Appendix A4 to compute the optimal expected revenue of the seller in the observable case (second column). In the third column, we present the expected revenue of our periodic pricing policy in the unobservable case. Finally, we compute the ratio between the latter values and the expected revenue of the best fixed price policy, 0.125, and present this values in the fourth column of the table. [3]

From the values in Table A1 we can see that, although our policy is better than fixed price, this advantage is small when the valuation is uniformly distributed.

In Table A2 we present a similar analysis for the TruncPareto(1,1,100) distribution. We remark here that this is possible because, in this particular case, we are able to explicitly solve the observable case, as shown in Appendix A4. Here, in contrast to the uniform case, we report that our policy performs significantly better than the fixed price policy, and the gap increases as the buyer's discount rate increases. Note that in this case the optimal fixed price policy gives revenue 0.5.[4]

[3] Note that the best fixed price policy for the uniform valuation, unobservable case is given by the price p maximizing $p(1-F(p)) = p(1-p)$, which is $p = 1/2$, and therefore the total expected revenue is $\frac{1}{4}\int_0^\infty e^{-t}e^{-t}\mathrm{d}t = \frac{1}{8}$, where one of the exponential factors in the integral comes from seller's discount rate and the other one from the density function of the exp(1) valuation distribution.

[4] Note that the best fixed price policy for the unobservable case under TruncPareto(1,1,M) valuations is given by the price p maximizing $p(1-F(p)) = \frac{1}{M-1}(M-p)$, which is $p = 1$, and thus the total expected revenue is $\int_0^\infty e^{-t}e^{-t} = \frac{1}{2}$.

Discount μ	Revenue observable case optimal policy	Revenue unobservable case periodic policy	Revenue ratio periodic policy vs. fixed price
1.1	0.1257	0.1232	0.9859
1.5	0.1287	0.1252	1.0016
2.0	0.1340	0.1274	1.0192
5.0	0.1563	0.1398	1.1184
7.0	0.1650	0.1439	1.1512
10.0	0.1741	0.1477	1.1816
100.0	0.2191	0.1463	1.1704

Table A1 Comparison among expected revenues under different buyer's discount rates μ, for Unif$[0,1]$ valuation distribution and exp(1) arrival distribution. Columns: Revenue from the optimal policy in the observable case (column 2), revenue from the periodic policy in the unobservable case (column 3), and ratio between the latter and that from the fixed price policy, i.e. 0.125, in the unobservable case (column 4). The seller discount rate is normalized to $\delta = 1$.

Discount μ	Revenue observable case optimal policy	Revenue unobservable case periodic policy	Revenue ratio periodic policy vs. fixed price
1.1	0.5079	0.4669	0.9338
1.5	0.5669	0.4932	0.9864
2.0	0.6358	0.5352	1.0704
5.0	0.9017	0.7328	1.4656
7.0	1.0098	0.8048	1.6096
10.0	1.1265	0.8667	1.7334
100.0	1.7841	1.1488	2.2976

Table A2 Comparison among expected revenues under different buyer's discount rates μ, for TruncPareto(1,1,M) valuation distribution and exp(1) arrival distribution. Columns: Revenue from the optimal policy in the observable case (column 2), revenue from the periodic policy in the unobservable case (column 3), and ratio between the latter and that from the fixed price policy, i.e. 0.5, in the unobservable case (column 4). The seller discount rate is normalized to $\delta = 1$.

A6. Unobservable case with truncated Pareto valuation and two possible arrival times

With the objective of characterizing the optimal solution of an unobservable arrival problem instance, suppose that the valuation of the buyer is distributed according to a TruncPareto(1,1,M), and that he arrives at one of two possible times: either at time 0 with probability β, or at time T with probability $1-\beta$, for some predetermined value $T > 0$.

We define the threshold valuation α as the value so that if the buyer arrives at time 0 with valuation $v \geq \alpha$, then he would buy before time T. This implies that, at time T, conditioned on the event that the seller has not sold the item yet, the buyer's valuation is the mixture of two truncated Pareto distributions: (i) a truncated Pareto in $[1,\alpha]$ accounting for the mass of buyers who arrived at 0 and decided to wait for a good price to be offered after T, with weight β, and (ii) a truncated Pareto in $[1,M]$ accounting for the buyer arriving at time T, with weight $1-\beta$.

Assume also that the seller's discount rate is normalized to $\delta = 1$, and that the buyer discounts the future at rate $\mu > 1$.

The general approach to compute the optimal expected revenue would be to decouple the problem in two independent subproblems that occur sequentially over time, by conditioning on the purchasing time of the buyer: either before or after T. Assume for now that the value of α is given. We will argue that if the buyer arrives at time zero with valuation above α, he does not have an incentive to delay his purchase beyond time T. Otherwise (i.e., he arrives at time zero with valuation below α or he arrives at time T), he will buy after time T. Hence, we can solve these two subproblems separately and then link them through the threshold α occurring at time T. Furthermore, each subproblem, once we condition on the information available at times 0 and T, corresponds to the observable case. In this regard, the threshold function ϕ of the unobservable arrival case is defined by parts combining three different purchasing functions ψ for the observable arrival case.

The first subproblem, defined over the time window $[0,T)$, is solved as in the observable case with TruncPareto(1,1,M) valuation but in a finite horizon, by adding a terminal condition for the purchasing function: $\psi(T) = \alpha$. For the second subproblem, defined over $[T,\infty)$, we first guess the time τ by which the buyer arriving at T would have purchased the item if and only if his valuation were above α, and then we solve two "observable arrival" problems assuming a truncated Pareto valuation distribution for each of them. More specifically, the problem in $[\tau,\infty)$ is solved with valuations TruncPareto(1,1,α); whereas the problem in the interval $[T,\tau)$ is solved with valuations TruncPareto(1,1,M) and with two terminal conditions: (i) the value of the purchasing time function at the boundary has to verify: $\psi(\tau) = \alpha$, and (ii) the price function has to be continuous at τ.

This whole procedure gives a price function and a purchasing function that depend on α and τ –see Figure A3–, which are then optimized to maximize the seller's revenue. Note that as we can derive explicit solutions for the truncated Pareto valuation distribution in the observable arrival case, the numerical part of the optimization to solve the unobservable case is only over these two parameters.

We provide below more details of the subproblems we need to solve to obtain the optimal price and threshold functions: one over the time interval $[0,T)$, and other over the time interval $[T,\infty)$, which in turn could be divided into two problems, with splitting time at τ, where in principle τ is assumed to be fixed. In particular, these three problems lead to three pricing policies, namely p_1, p_2 and p_3, with corresponding purchasing functions ψ_1, ψ_2 and ψ_3, as can be seen in Figure A3. After that, we prove that the problem can indeed be decoupled into these sub-problems, by showing that there are no purchasing deviations (see Propositions A3 and A4 below).

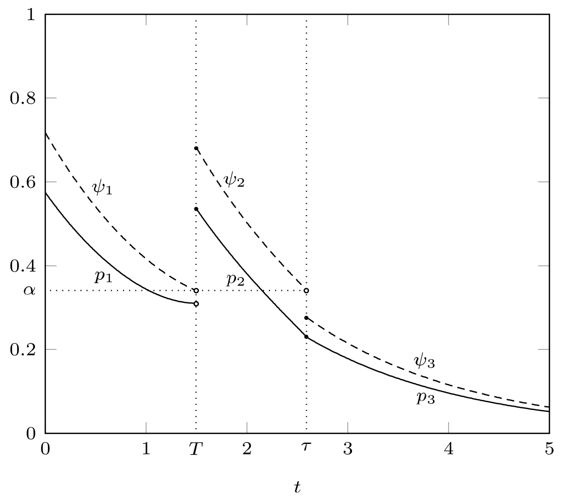


Figure A3 Price and purchasing functions for the unobservable arrival case with TruncPareto(1,0,1) and two possible arrival times: 0 and T. The threshold function ϕ is defined by three parts through respective definitions of the purchasing function ψ for the observable arrival case. In order to compel with the requirement of ϕ being lower semicontinuous, the function ψ_1 is assumed to be left continuous at some point $T-\epsilon$, for $\epsilon > 0$ arbitrarily small, and the function ψ_2 is extended to the left of T so that it is continuous at T. Note that this technical adjustment implies a negligible revenue loss from the seller's perspective.

The following steps are performed for solving a particular problem instance, defined over the parameters β, T, μ, and M. The procedure is described for fixed given values α and τ, leading to a pricing policy p and a threshold function ϕ that depend on both values. Then, both α and τ are optimized to maximize the revenues of the specific problem instance.

Step 1. Compute the pricing policy p_1 to offer in the time interval $[0,T)$ and its associated purchasing function ψ_1. To this end, we need to solve the problem $[SPO_0^r]$ affected by the probability β that the buyer indeed arrives at time 0, i.e.,

$$\max_p \quad \beta p_1(0)(1-F(\psi_1(0))) + \beta \int_0^T e^{-t} p_1(t) f(\psi_1(t))(-\psi_1'(t))\,\mathrm{d}t,$$

where $\psi_1(t)$ is defined in (1), and where we further impose the boundary condition $\psi_1(T)=\alpha$.

Step 2A. Compute the pricing policy p_3 to offer in the time interval $[\tau,\infty)$ and its associated purchasing function ψ_3. To this end, we need to solve the problem $[SPO_0^r]$ but with origin at time τ, with valuations at most α, i.e.,:

$$\max_p \int_\tau^\infty e^{-t} p_3(t) f(\psi_3(t))(-\psi_3'(t))\,\mathrm{d}t + e^{-\tau} p_3(\tau)(F(\alpha)-F(\psi_3(\tau))),$$

where the second term represents the discounted expected revenue obtained at time τ from the mass of valuations between $\psi_3(\tau)$ and α who arrived at time 0 or at time T,[5] and the term involving the integral represents the discounted expected revenue obtained along the interval $[\tau, \infty)$.

Step 2B. Compute the pricing policy p_2 to offer in the time interval $[T, \tau)$ and its associated purchasing function ψ_2. To this end, we need to solve the problem $[SPO_0^r]$ affected by the probability $1-\beta$ that the buyer arrives at time T, i.e.,

$$\max_{p} \quad (1-\beta)e^{-T}p_2(T)(1-F(\psi_2(T))) + (1-\beta)\int_T^\tau e^{-t}p_2(t)f(\psi_2(t))(-\psi_2'(t))\,\mathrm{d}t,$$

where $\psi_2(t)$ is defined in (1), and where we further impose two boundary conditions: $\psi_2(\tau) = \alpha$, and $p_2(\tau) = p_3(\tau)$.[6]

With the solutions of the three steps above, we can define the pricing policy p and the threshold function ϕ for the whole horizon. To ensure that these functions solve the unobservable case for this particular instance, it only remains to prove that if the buyer arrives at time 0 with valuation $v > \alpha$, then he will indeed buy before time T. To this end, we show a preliminary result stating that if a buyer with valuation $\psi_1(\eta)$ for some $\eta \in [0, T)$ buys after T, then he will also buy after T if he has valuation belonging to the interval $(\alpha, \psi_1(\eta))$, where ψ_1 represents the solution of the problem described in Step 1.

PROPOSITION A3. *If there exists $\eta \in [0, T)$ and $\eta' > T$ such that $U(\eta, \psi_1(\eta)) < U(\eta', \psi_1(\eta))$, then for all t belonging to the interval $(\eta, T]$, there exists $t' > T$ such that $U(t, \psi_1(t)) < U(t', \psi_1(t))$.*

Proof. By contradiction, suppose that there exist $t \in (\eta, T]$ such that $U(t, \psi_1(t)) \geq U(t', \psi_1(t))$ for all $t' > T$. In particular, the inequality holds for $t' = \eta'$, and then we have $U(t, \psi_1(t)) \geq U(\eta', \psi_1(t))$.

Define $\varepsilon = \psi_1(\eta) - \psi_1(t)$. Note that by hypothesis $\eta < t$, and thus, from the monotonicity of ψ_1 (see Proposition 1 in Section 4), we obtain $\varepsilon \geq 0$, which implies

$$U(t, \psi_1(\eta)) = e^{-\mu t}(\psi_1(t) + \varepsilon - p(t)) = U(t, \psi_1(t)) + \varepsilon e^{-\mu t} \geq U(\eta', \psi_1(t)) + \varepsilon e^{-\mu\eta'} = U(\eta', \psi_1(t)),$$

where the inequality holds due to the contradiction hypothesis for $t' = \eta'$ and because $\eta' > t$, and the last equality holds because $\psi_1(\eta) = \psi_1(t) + \varepsilon$. We conclude that $U(t, \psi_1(\eta)) \geq U(\eta', \psi_1(\eta))$.

[5] This mass of valuations is composed of $\beta(F(\alpha) - F(\psi_3(\tau)))$ coming from time 0, and $(1-\beta)(F(\alpha) - F(\psi_3(\tau)))$ coming from time T. Recall that no matter the arrival time, the buyer always discounts his utility since time 0.

[6] Note that the price function has to be continuous at τ. Otherwise, knowing that p is non increasing (due to the observation in Section 3.2.1 together with Proposition 2), if it were the case that $\lim_{t\to\tau^-} p_2(\tau) > p_3(\tau)$, then a buyer with valuation $\psi_2(\tau - \epsilon) > \alpha$ would have an incentive to wait and buy at τ and take advantage of a price decreased by a non negligible amount, which would contradict the definition of α as the threshold valuation above which a buyer would purchase before τ.

On the other hand, by hypothesis we know that $U(\eta', \psi_1(\eta)) > U(\eta, \psi_1(\eta))$, and putting all together we conclude that $U(t, \psi_1(\eta)) > U(\eta, \psi_1(\eta))$ which contradicts the definition of the purchasing function ψ_1, completing the proof. □

Let us use this result to prove that if the buyer arrives at time 0 with valuation greater than α, then he does not have an incentive to buy later than T. This result justifies the decoupling of the problem that occurs at time T and that separates Step 1 from Step 2.

PROPOSITION A4. *If the buyer arrives at time* 0 *with valuation* $v > \alpha$*, then there exists an optimal pricing policy under which he purchases at some time before* T*.*

Proof. By contradiction, suppose that there exist $t \in (\eta, T]$ such that $U(t, \psi_1(t)) \geq U(t', \psi_1(t))$ for all $t' > T$. In particular, the inequality holds for $t' = \eta'$, and then we have $U(t, \psi_1(t)) \geq U(\eta', \psi_1(t))$.

Define $\varepsilon = \psi_1(\eta) - \psi_1(t)$. Note that by hypothesis $\eta < t$, and thus, from the monotonicity of ψ_1 (see Proposition 1 in Section 4), we obtain $\varepsilon \geq 0$, which implies

$$U(t, \psi_1(\eta)) = e^{-\mu t}(\psi_1(t) + \varepsilon - p(t)) = U(t, \psi_1(t)) + \varepsilon e^{-\mu t} \geq U(\eta', \psi_1(t)) + \varepsilon e^{-\mu \eta'} = U(\eta', \psi_1(t)),$$

where the inequality holds due to the contradiction hypothesis for $t' = \eta'$ and because $\eta' > t$, and the last equality holds because $\psi_1(\eta) = \psi_1(t) + \varepsilon$. We conclude that $U(t, \psi_1(\eta)) \geq U(\eta', \psi_1(\eta))$.

On the other hand, by hypothesis we know that $U(\eta', \psi_1(\eta)) > U(\eta, \psi_1(\eta))$, and putting all together we conclude that $U(t, \psi_1(\eta)) > U(\eta, \psi_1(\eta))$ which contradicts the definition of the purchasing function ψ_1, completing the proof. □

In summary, even though the unobservable arrival case is hard to solve in general, in the particular scenario with truncated Pareto distribution and two possible arrival times, we could solve it by simplifying its formulation to a sequence of observable arrival cases. We highlight here that the same argument can be used to solve the unobservable case with two arrival times for any setting for which the observable case can be solved, as it is the case of the uniform distribution.

A7. A lower bound for the value of observability

In order to get a lower bound for the value of observability it is enough to get the ratio for one particular problem instance. The challenge here stems from the difficulty in solving the unobservable case, even numerically. In order to partially overcome this difficulty, we consider the particular problem instance introduced in Appendix A6: TruncPareto$(1, 1, M)$ valuations and two possible arrival times: 0 and T, with probability β and $1 - \beta$, respectively. Remember that the policy is further characterized by two additional parameters: (i) α, representing the value so that if the buyer arrives at time 0 with valuation $v \geq \alpha$, then he would buy before time T, and (ii) τ, the time by which a buyer arriving at T would have purchased the item if and only if his valuation were above α.

Setting the parameters $M = 100$ and $T = 1$, and normalizing the discount rate of the seller at $\delta = 1$, we numerically compute the value of observability for different buyer's discount rates μ and probabilities β of arriving at time 0. The results are exhibited in Table A3, from where we note that the worst case over these instances is the one determined by $\mu = 4.5$ and $\beta = 0.4$, leading to a lower bound of 1.136 for the value of observability.

μ \ β	0.2	0.4	0.6	0.8
1.0	1	1	1	1
1.5	1.036	1.017	1.005	1.001
2.0	1.074	1.047	1.020	1.001
2.5	1.111	1.069	1.038	1.004
3.0	1.128	1.089	1.048	1.013
3.5	1.124	1.107	1.058	1.020
4.0	1.115	1.124	1.066	1.025
4.5	1.103	**1.136**	1.074	1.029
5.0	1.091	1.121	1.094	1.064
5.5	1.078	1.103	1.075	1.044
6.0	1.065	1.084	1.054	1.022

Table A3 Value of Observability for a two-point arrival time distribution: 0 with probability β, and $T = 1$ with probability $1 - \beta$, assuming TruncPareto(1,1,100) valuation and seller's discount rate $\delta = 1$.